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ON THE LOCAL REGULARITY OF SOLUTIONS FOR DOUBLE DEGENERATE NONLINEAR PARABOLIC EQUATIONS $(u^{q-1})_t = \text{div}\,(|\nabla u|^{p-2}\nabla u)$ WHEN $1 < p < 2, p \le q$*

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Dedicated to the 70th birthday of Professor Zhou Yulin

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Abstract In this paper, we establish interior Hölder estimates of solutions for double degenerate nonlinear parabolic equations $(u^{q-1})_t = \text{div}\,(|\nabla u|^{p-2}\nabla u)$ when $1 < p < 2$, $p \le q$.

Key Words Hölder continuity; double degenerate; nonlinear parabolic equations
Classifications 35K55, 35K65

1. Introduction

In this paper, we are mainly concerned with local Hölder continuity of nonnegative weak solution for the following double degenerate parabolic equations

$$(u^{q-1})_t = \text{div}\,(|\nabla u|^{p-2}\nabla u) \quad \text{in } Q_T \tag{1.1}$$

where $1 < p < 2$, $p \le q$, $Q_T = \Omega \times (0,T]$, Ω is an open set in $R^N (N \ge 1)$, $\nabla u = \left(\frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_N}\right)$.

For $p = 2$, (1.1) may be considered as the porous media equations $v_t = \Delta(v^{\frac{1}{q-1}})$ with $v = u^{q-1}$. Hölder continuity of solutions for porous media equations was proven in past, see [1], [2], [6], [7].

When $q = 2$, (1.1) is evolutionary p-Laplace equation, Hölder estimates for its weak solution and gradients of solutions have recently been obtained, see [3]-[6].

For double degenerate equations (1.1), the existence and uniqueness theorem and other properties of solutions have recently been investigated by some works, see [10]-[12]. When $1 < q \le p$, $p > 2$, Hölder continuity of solutions of (1.1) has just been proven by the authors, see [8].

For a weak solution u (supersolution, subsolution) of (1.1), we mean that $u \ge 0$, $u \in L^p(0,T;W^{1,p}(\Omega))$, $v, v_t \in L^2(Q_T)$, where $v = u^{q-1}$, and u satisfies

$$\int_{t_1}^{t_2}\int_\Omega v_t \varphi \, dx dt = (\ge, \le) \int_{t_1}^{t_2}\int_\Omega |\nabla u|^{p-2}\nabla u \nabla\varphi \, dx dt \tag{1.2}$$

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for $0 \le t_1 < t_2 \le T$, $\varphi \in L^2(Q_T) \cap L^p(0,T; W_0^{1,p}(\Omega)), \varphi \ge 0$.

Under appropriate conditions, one can prove the local boundedness of weak solution for (1.1). Throughout this paper we assume $0 \le u \le M$.

Our main result is the following.

Theorem 1.1 *Assume that u is a weak solution of (1.1) with $1 < p < 2$, $p \le q$, and $0 \le u \le M$. Then for any $\varepsilon \in (0,1)$, there exist constants β, $C > 0$ dependent only on p, q, N, M, ε, $0 < \beta < 1$, such that*

$$|u(x_1,t_1) - u(x_2,t_2)| \le C(|x_1 - x_2| + |t_1 - t_2|^{1/p})^\beta$$

for all (x_1,t), $(x_2,t_2) \in \Omega_\varepsilon \times (\varepsilon, T-\varepsilon)$, $\Omega_\varepsilon = \left\{x \in \Omega : |x| < \frac{1}{\varepsilon}, d(x,\partial\Omega) > \varepsilon\right\}$.

2. Preliminary

In this section, we will give several Lemmas used later. Set

$$K_R(x_0) = \{x \in R^N : |x^i - x_0^i| \le R,\ 1 \le i \le N\}$$
$$Q(R,\rho;z_0) = K_R(x_0) \times (t_0 - \rho, t_0], z_0 = (x_0,t_0)$$

Assume $Q(R,\rho;z_0) \subset Q_T$.

Lemma 2.1 *If u is a supersolution of (1.1), then*

$$\begin{aligned}\sup_{t_0-\rho<t\le t_0} &\int_{K_R(x_0)} \zeta^p \Big[\int_u^k s^{q-2}(k-s)^+ ds\Big] dx + \iint_{Q(R,\rho;z_0)} \zeta^p |\nabla(k-u)^+|^p dxdt \\ &\le C \iint_{Q(R,\rho;z_0)} \Big\{|\nabla\zeta|^p (k-u)^{+p} + \zeta^{p-1}|\zeta_t| \Big[\int_u^k s^{q-2}(k-s)^+ ds\Big]\Big\} dxdt \quad (2.1)\end{aligned}$$

If u is a subsolution of (1.1), then

$$\begin{aligned}\sup_{t_0-\rho<t\le t_0} &\int_{K_R(x_0)} \zeta^p \Big[\int_k^u s^{q-2}(s-k)^+ ds\Big] dx + \iint_{Q(R,\rho;z_0)} \zeta^p |\nabla(u-k)^+|^p dxdt \\ &\le C \iint_{Q(R,\rho;z_0)} \Big\{|\nabla\zeta|^p (u-k)^{+p} + \zeta^{p-1}|\zeta_t| \Big[\int_k^u s^{q-2}(s-k)^+ ds\Big]\Big\} dxdt \quad (2.2)\end{aligned}$$

In (2.1) and (2.2), $k > 0$, constant c depends only on p,q. $\zeta \ge 0$, $\zeta \in C^1(\bar{Q}(R,\rho;z_0))$, $\zeta|_{\partial_p Q(R,\rho;z_0)} = 0$.

Proof In (1.2), by taking $\varphi = \zeta^p(k-u)^+$, we easily obtain (2.1). Similarly, (2.2) can be proven.

Lemma 2.2 *For $1 < p < 2$, $q \ge p$, there exist C_1, C_2 dependent only on q, p such that for $u \ge 0$*

$$C_1 k^{q-2}(k-u)^{+2} \le \int_u^k s^{q-2}(k-s)^+ ds \le C_2 k^{q-p}(k-u)^{+p} \quad (2.3)$$

and for $0 < \frac{\mu}{2} \le k \le u \le \mu$

$$C_1\mu^{q-2}(u-k)^{+2} \le \int_k^u s^{q-2}(s-k)^+ ds \le C_2\mu^{q-2}(u-k)^{+2} \tag{2.4}$$

Proof Clearly

$$\int_u^k s^{q-2}(k-s)^+ ds \ge \int_{\frac{k+u}{2}}^k s^{q-2}(k-s)^+ ds \ge C_1 k^{q-2}(k-u)^{+2}$$

On the other hand, by mean value theorem, obviously

$$\begin{aligned}\int_u^k s^{p-2}(k-s)^+ ds &= \frac{1}{p-1}\int_{u^{p-1}}^{k^{p-1}} (k - s^{\frac{1}{p-1}})^+ ds = \frac{1}{p-1}\int_{u^{p-1}}^{k^{p-1}} ((k^{p-1})^{\frac{1}{p-1}} - s^{\frac{1}{p-1}})ds \\ &\le \int_{u^{p-1}}^{k^{p-1}} k^{2-p}(k^{p-1}-s)^+ ds \le \frac{1}{2}k^{2-p}(k-u)^{+2(p-1)}\end{aligned} \tag{2.5}$$

Therefore for $0 \le u < \frac{k}{2}$, from (2.5), we have

$$\int_u^k s^{p-2}(k-s)^+ ds \le \frac{1}{2}2^{2-p}(k-u)^{+p}$$

For $\frac{k}{2} \le u < k$, we also have

$$\int_u^k s^{p-2}(k-s)^+ ds \le \frac{1}{2}2^{2-p}(k-u)^{+p}$$

Thus, it follows that

$$\int_u^k s^{q-2}(k-s)^+ ds \le \frac{1}{2}2^{2-p}k^{q-p}\frac{1}{2}(k-u)^{+p}$$

Therefore (2.3) follows. It is easy to check (2.4).

For $0 \le s \le H$, define

$$\Psi(H,s,\nu) = \ln^+ \frac{H}{H-s+\nu}, \quad 0 < \nu < \min\{1,H\}$$

Lemma 2.3 *Assume u is a subsolution of* (1.1), *then*

$$\begin{aligned}&\int_{K_R(x_0)} \zeta^p(x)\Big[\int_k^u s^{q-2}(\Psi^2)'(H,(s-k)^+,\nu)ds\Big]\Big|_{t_0-\rho}^t dx \\ &\le C\int_{t_0-\rho}^t \int_{K_R(x_0)} |\nabla\zeta|^p \Psi(\Psi')^{2-p}(H,(u-k)^+,\nu)dxdt\end{aligned} \tag{2.6}$$

where $t_0-\rho < t \le t_0$, $H \ge \sup_{Q(R,\rho;x_0)}(u-k)^+$, $0 \le \zeta \in C_0^1(K_R(x_0))$, *$C$ depends only on* p, q.

Proof Take $\varphi = \zeta^p(x)(\Psi^2)'(H,(u-k)^+,\nu)$ in (1.2), note $(\Psi^2)'' = 2(1+\Psi)(\Psi')^2$, it is easy to get (2.6).

For the following two Lemmas, refer to [9], [5].

Lemma 2.4 (De Giorgi) *If $f \in W^{1,1}(K_R(x_0))$ and $l, k \in R$, $l > k$, then*

$$(l-k)\text{meas } A^+_{l,R} \le \frac{CR^{N+1}}{\text{meas }[K_R \setminus A_{k,R}]} \int_{A^+_{k,R} \setminus A^+_{l,R}} |\nabla f| dx$$

where $A^+_{l,R} = \{x \in K_R(x_0) : f(x) > l\}$, and c depends only on N.

Lemma 2.5 *If $u \in \overset{\circ}{V}_p(Q_T) = L^\infty(0,T;L^p(\Omega)) \cap L^p(0,T;W^{1,p}_0(\Omega))$, then*

$$\|u\|_{L^{p^*}(Q_T)} \le C\|u\|_{\overset{\circ}{V}_p(Q_T)}$$

where $p^* = p\dfrac{N+p}{N}$, $\|u\|^p_{\overset{\circ}{V}_p(Q_T)} = \sup\limits_{0<t<T} \|u(\cdot,t)\|^p_{L^p(\Omega)} + \|\nabla u\|^p_{L^p(Q_T)}$; C depends only on N, p.

Let $Q^A_R(x_0,t_0) = K_{AR}(x_0) \times (t_0 - R^p, t_0]$

Corollary 2.6 *Assume $u \in \overset{\circ}{V}_p(Q^A_R(x_0,t_0))$. Then*

$$\fint\!\!\fint_{Q^A_R(x_0,t_0)} |u|^p dxdt \le C\frac{(AR)^p}{|Q^A_R(x_0,t_0)|}\Big(\fint\!\!\fint_{Q^A_R(x_0,t_0)} \chi[u \ne 0] dxdt\Big)^{p/(N+p)}$$
$$\cdot\Big(\sup_{t_0 - R^p < t \le t_0} \frac{1}{A^p}\int_{K_{AR}(x_0)} |u|^p dx + \iint_{Q^A_R(x_0,t_0)} |\nabla u|^p dxdt\Big)$$

where $\chi\Big[\sum\Big]$ denotes the characteristic function of $\sum$, $\fint\!\!\fint_\Sigma f dxdt = \dfrac{1}{|\sum|}\iint_\Sigma f dxdt$, and C depends only on N, p.

Proof By transformation

$$x = x_0 + 2AR\tilde{x}, t = t_0 - R^p\tilde{t}; \tilde{u}(\tilde{x},\tilde{t}) = u(x_0 + 2AR\tilde{x}, t_0 - R^p\tilde{t})$$

we have $\fint\!\!\fint_{Q^A_R(x_0 t_0)} |u|^p dxdt = \iint_{K_{\frac{1}{2}}(0)\times(-1,0]} |\tilde{u}|^p d\tilde{x}d\tilde{t}$, then Lemma 2.5 implies Corollary 2.6.

The following Lemma is the direct generalization of Lemma 3.2 of [4].

Lemma 2.7 *If u_1, u_2 are both supersolutions of (1.1), then $\omega = \min\{u_1,u_2\}$ is also a supersolution of (1.1).*

Proof By (1.2), it follows that

$$\int_{t_1}^{t_2}\int_\Omega \{(u_i^{q-1})_t\varphi_i + |\nabla u_i|^{p-2}\nabla u_i \nabla\varphi_i\} dxdt \ge 0 \tag{2.7}$$

for $\varphi_i \in L^2(Q_T) \cap L^p(0,T;W^{1,p}_0(\Omega))$, $\varphi_i \ge 0$, $i = 1,2; 0 \le t_1 < t_2 \le T$. For $\varepsilon \in (0,1]$ and $0 \le \varphi \in L^2(Q_T) \cap L^p(0,T;W^{1,p}_0(\Omega))$, define

$$H_\varepsilon(s) = 0 \text{ for } s \le 0,\ H_\varepsilon(s) = \frac{s}{\varepsilon} \text{ for } 0 < s \le \varepsilon,\ H_\varepsilon(s) = 1 \text{ for } s > \varepsilon$$

$$\varphi_1 = H_\varepsilon(u_2 - u_1)\cdot\varphi,\ \varphi_2 = [1 - H_\varepsilon(u_2 - u_1)]\varphi$$

Obviously, $H_\varepsilon(u_2-u_1)\to\chi[u_2>u_1]$ as $\varepsilon\to 0$.

$$\int_{t_1}^{t_2}\int_\Omega\sum_{i=1}^{2}(u_i^{q-1})_t\varphi_i dxdt$$
$$+\int_{t_1}^{t_2}\int_\Omega[|\nabla u_1|^{p-2}\nabla u_1 H_\varepsilon(u_2-u_1)+|\nabla u_2|^{p-2}\nabla u_2(1-H_\varepsilon(u_2-u_1))]\nabla\varphi dxdt$$
$$\geq -\int_{t_1}^{t_2}\int_\Omega[|\nabla u_1|^{p-2}\nabla u_1-|\nabla u_2|^{p-2}\nabla u_2]H_\varepsilon'(u_2-u_1)\nabla(u_2-u_1)\varphi dxdt$$

Let $\varepsilon\to 0$, it is easy to get

$$\int_{t_1}^{t_2}\int_\Omega\{(\omega^{q-1})_t\varphi+|\nabla\omega|^{p-2}\nabla\omega\nabla\varphi\}dxdt$$
$$\geq\varliminf_{\varepsilon\to 0}\int_{t_1}^{t_2}\int_\Omega(|\nabla u_2|^{p-2}\nabla u_2-|\nabla u_1|^{p-2}\nabla u_1)(\nabla u_2-\nabla u_1)H_\varepsilon'(u_2-u_1)\varphi dxdt\geq 0$$

Therefore ω is a supersolution of (1.1).

Now we introduce some notations. Set

$$K_R^\theta(x_0)=\{x\in R^N:|x^i-x_0^i|\leq\theta R, 1\leq i\leq N\},\ \theta=2^{s^*(2-p)/p}\mu^{(p-q)/p}$$
$$Q_R^\theta(x_0,t_0)=K_R^\theta(x_0)\times(t_0-R^p,t_0]$$
$$Q_R^\eta(x_0,t_0)=K_R^\theta(x_0)\times(t_0-R^p,t_0],\ \eta=\mu^{(p-q)/p}$$

where $\mu>0$, s^* dependent only on p,q,N will be determined in Lemma 4.3 and chosen, such that $2^{s^*(2-p)/p}$ is an integer. For convenience, denote $K_R(0)$ by K_R and $Q_R^\theta(0,0)$ by Q_R^θ, etc.. Denote various constants dependent on given data by C.

In the following, let $\mu^-=\inf\limits_{Q_{4R}^\theta}u$ and assume that $Q_{4R}^\theta\subset Q_T$, and $\sup\limits_{Q_{4R}^\theta}u\leq\mu_1$, which can be insured, for instance, take $\mu=M$.

Since (1.1) is degenerate at $u=0$ or $\nabla u=0$, the proof of Hölder continuity of its solution is more complicated. Our method is firstly to fine some subbox of Q_{4R}^θ in which $\sup u$ is smaller than μ or $\inf u$ is positive. Because of the double degenerate, we will work with Q_{4R}^θ instead of working with other shaped cylinders or boxes used in [2]–[9]. We will consider either of the following two cases, respectively in Sections 3 and 4.

Case I There exists some subbox $Q_{2R}^\eta(\bar{x},0)\subset Q_{2R}^\theta$ such that

$$\text{meas}\left\{(x,t)\in Q_{2R}^\eta(\bar{x},0):u(x,t)<\mu^-+\frac{\mu}{4}\right\}\leq\alpha_0|Q_{2R}^\eta|$$

Case II For all subbox $Q_{2R}^\eta(\bar{x},0)\subset Q_{2R}^\theta$

$$\text{meas}\left\{(x,t)\in Q_{2R}^\eta(\bar{x},0):u(x,t)\geq\mu^-+\frac{\mu}{4}\right\}<(1-\alpha_0)|Q_{2R}^\eta|$$

here α_0 is some constant dependent only on p,q,N.

Secondly, in Section 5 we will construct a series of subboxes $Q^{\theta_i}_{\mu_i}$ which shrink to some point $(0,\tau_0)$ such that either $u(0,\tau_0)=0$ and $\sup\limits_{Q^{\theta_i}_{\mu_i}} u = 0$ (as $i\to\infty$), or for some $Q^{\theta_i}_{\mu_i}$, $\inf\limits_{Q^{\theta_i}_{\mu_i}} u > 0$. In later case, (1.1) can be considered as evolutionary p-Laplace equation which is only a single degenerate.

3. Discussion in Case I

In this section we assume that u is a supersolution of (1.1), and that there exists some subbox $Q^{\eta}_{2R}(\bar{x},0)\subset Q^{\theta}_{2R}$ such that

$$\text{meas}\left\{(x,t)\in Q^{\eta}_{2R}(\bar{x},0): u(x,t)<\mu^- + \frac{\mu}{4}\right\}\le \alpha_0|Q^{\eta}_{2R}| \tag{3.1}$$

where α_0 will be determined in the proof of Lemma 3.1.

Lemma 3.1 *Suppose (3.1) holds. If α_0 is sufficiently small and $0<\mu^-<\frac{\mu}{8}$, then*

$$u(x,t)\ge \mu^- + \frac{\mu}{8},\ \textit{for}\ (x,t)\in Q^{\eta}_{R}(\bar{x},0) \tag{3.2}$$

Proof For $i=0,1,2,\cdots$, set

$$k_i=\mu^- + \frac{\mu}{8}+\frac{\mu}{2^{3+i}};\quad R_i = R+\frac{R}{2^i}$$

$$Q_i = K^{\eta}_{R_i}(\bar{x})\times(-R_i^p,0]$$

Clearly for $i\ge 0$, $\frac{\mu}{8}\le k_i\le\frac{\mu}{2}$. By (2.1), it is easy to get

$$\sup_{-R_i^p<t\le 0}\int_{K^{\eta}_{R_i}(\bar{x})}\zeta^p\mu^{q-2}(k_i-u)^{+2}dx + \iint_{Q_i}\zeta^p|\nabla(k_{i+1}-u)^+|^p dxdt$$
$$\le C\iint_{Q_i}\{|\nabla\zeta|^p(k_i-u)^{+p}+\zeta^{p-1}|\zeta_t|\mu^{q-p}(k_i-u)^{+p}\}dxdt \tag{3.3}$$

Take ζ as a cutoff function in Q_i, by Corollary 2.6, it follows that

$$\iint_{Q_{i+1}}(k_{i+1}-u)^{+p}dxdt\le C\frac{(\eta R_i)^p}{|Q_i|}\Big(\iint_{Q_i}\chi[u<k_{i+1}]dxdt\Big)^{p/(N+p)}$$
$$\cdot\Big(\sup_{-R_i^p<t\le 0}\frac{1}{\eta^p}\int_{K^{\eta}_{R_i}(\bar{x})}\zeta^p(k_{i+1}-u)^{+p}dx+\iint_{Q_i}\{|\nabla\zeta|^p(k_{i+1}-u)^{+p}$$
$$+\zeta^p|\nabla(k_{i+1}-u)^+|^p\}dxdt\Big) \tag{3.4}$$

Notice $\eta=\mu^{(p-q)/p}$, $\mu^-=\inf\limits_{Q^{\theta}_{4R}} u$, obviously

$$\frac{1}{\eta^p}\int_{K^{\eta}_{R_i}(\bar{x})}\zeta^p(k_{i+1}-u)^{+p}dx\le 2^{(i+4)(2-p)}\mu^{q-2}\int_{K^{\eta}_{R_i}(\bar{x})}\zeta^p(k_i-u)^{+2}dx \tag{3.5}$$

Similarly, by (3.3), we can also estimate $\iint_{Q_i} \zeta^p|\nabla(k_{i+1}-u)^+|^p dxdt$. Combining this estimate with (3.4), (3.5), we get

$$\begin{aligned}&\iint_{Q_{i+1}} (k_{i+1}-u)^{+p}dxdt\\ &\le C\frac{(\eta R_i)^p}{|Q_i|}2^i\Big(\iint_{Q_i}\frac{(k_i-u)^{+p}}{(k_i-k_{i+1})}dxdt\Big)^{\frac{p}{N+p}}\\ &\cdot\iint_{Q_i}\{|\nabla\zeta|^p(k_i-u)^{+p}+\zeta^{p-1}|\zeta_t|\mu^{q-p}(k_i-u)^{+p}\Big\}dxdt\end{aligned}$$

Take ζ so that $|\nabla\zeta|\le\dfrac{C2^i}{\eta R}$, $|\zeta_t|\le\dfrac{C2^i}{R^p}$, since $R\le R_i\le 2R$, we have

$$\begin{aligned}&\frac{1}{\mu^p}\iint_{Q_{i+1}}(k_{i+1}-u)^{+p}dxdt\\ &\le C2^{3p_i}\Big(\frac{1}{\mu^p}\iint_{Q_i}(k_i-u)^{+p}dxdt\Big)^{1+[p/(N+p)]}\end{aligned}$$

On the other hand, from (3.1), it is easy to check

$$\frac{1}{\mu^p}\iint_{Q_0}(k_i-u)^{+p}dxdt\le\Big(\frac{1}{4}\Big)^p\alpha_0$$

If α_0 is sufficiently small, according to Lemma 5.7 of [9], (3.2) follows.

Remark Observe that in the proof of Lemma 3.1, the condition $\mu\ge\sup_{Q^\theta_{4R}} u$ is not necessary. We only use the assumption $\eta=\mu^{(p-q)/p}\le\theta$.

Next we will expand the set where u is "large" sidewise in the spatial direction up to a small box around (0, 0). The process for the case $p=q$ is simpler than for the case $p<q$, hence we discuss these two cases, respectively.

Let $Q^\theta_R(h_0)=K^\theta_{2R}(0)\times(-R^p,-(1-h_0)R^p]$, $h_0\in(0,1]$. For any finite number $h_j(1\le j\le m)$, $-(1-h_0)+h_j\le 0$, set

$$\omega(x,t)=\min_{1\le j\le m}\{u(x,t+h_jR^p)\}$$

According to Lemma 2.7, ω is also a supersolution of (1.1). Since $K^\eta_R(\bar{x})\subset K^\theta_{2R}$, by (3.2), we have

$$\text{meas}\Big\{x\in K^\theta_{2R}:\omega(x,t)\ge\mu^-+\frac{\mu}{8}\Big\}\ge 2^{Ns^*\frac{(p-2)}{p}-N}|K^\theta_{2R}| \tag{3.6}$$

for all $t\in(-R^p,-(1-h_0)R^p]$.

Lemma 3.2 *If ω satisfies (3.6) and $0\le\mu^-<\dfrac{\mu}{8}$, then for any $\alpha_1>0$, there exists s_* dependent only on p,q,N,s^*,h_0,α, such that*

$$meas\Big\{(x,t)\in Q^\theta_R(h_0):\omega(x,t)<\mu^-+\frac{\mu}{2^{s_*}}\Big\}\le\alpha_1|Q^\theta_R(h_0)|$$

Proof Let $k_i = \mu^- + \frac{\mu}{2^i}, i = 3, \cdots, s_*$. In Lemma 2.4, take $f = -\omega$, $l = -k_{i+1}$, $k = -k_i$, then

$$\frac{\mu}{2^{i+1}} \text{meas } A^-_{k_{i+1},\theta,2R}(t) \leq \frac{C(2\theta R)^{N+1}}{\text{meas } (K^\theta_{2R} \setminus A^-_{k_i,\theta,2R}(t))} \int_{A^-_{k_i,\theta,2R}(t)\setminus A^-_{k_{i+1},\theta,2R}(t)} |\nabla \omega| dx \quad (3.7)$$

where $A^-_{k,\theta,2R}(t) = \{x \in K^\theta_{2R} : \omega(x,t) < k\}$.

Integrate (3.7) with respect to t, denote $A_i = \{(x,t) \in Q^\theta_R(h_0) : \omega(x,t) < k_i\}$, by (3.6) and Hölder's inequality, it follows that

$$\begin{aligned} \frac{\mu}{2^{i+1}} | A_{i+1}| &\leq C 2^{s^*(2-p)(N+1)/p} \mu^{(p-q)/p} R \Big(\iint_{A_i} |\nabla \omega|^p dxdt \Big)^{\frac{1}{p}} |A_i - A_{i+1}|^{(p-1)/p} \\ &= C 2^{s^*(2-p)(N+1)/p} \mu^{(p-q)/p} R \Big(\iint_{Q^\theta_R(h_0)} |\nabla (k_i - \omega)^+|^p dxdt \Big)^{\frac{1}{p}} |A_i - A_{i+1}|^{\frac{p-1}{p}} \end{aligned} \quad (3.8)$$

On the other hand, by (2.1), we have

$$\begin{aligned} &\iint_{\tilde{Q}^\theta_{2R}(x_0)} \zeta^p |\nabla (k_i - \omega)^+|^p dxdt \\ &\leq C \iint_{\tilde{Q}^\theta_{2R}(h_0)} \Big\{ |\nabla \zeta|^p (k_i - \omega)^{+p} + \zeta^{p-1} |\zeta_t| \Big[\int_\omega^{k_i} s^{q-2} (k_i - s)^+ ds \Big\} dxdt \end{aligned} \quad (3.9)$$

where $\tilde{Q}^\theta_{2R}(h_0) = K^\theta_{4R} \times (-(1+h_0)R^p, -(1-h_0)R^p]$. Take cutoff function ζ so that $\zeta = 1$ in $Q^\theta_R(h_0)$, $|\nabla \zeta| \leq \frac{C}{\theta R}$, and $|\zeta_t| \leq \frac{c}{h_0 R^p}$, by (3.8), (3.9) and (2.3), noting $|Q^\theta_R(h_0)| = 2^{-(N+1)} |\tilde{Q}^\theta_{2R}(h_0)|$, we obtain

$$\begin{aligned} \frac{\mu}{2^{i+1}} |A_{i+1}| &\leq C \mu^{(p-q)/p} R \Big(\frac{1}{(\theta R)^p} \Big(\frac{\mu}{2^i} \Big)^p \\ &\quad + \frac{1}{h_0 R^p} k_i^{q-p} \Big(\frac{\mu}{2^i} \Big)^p \Big)^{\frac{1}{p}} |Q^\theta_R(h_0)|^{1/p} |A_i - A_{i+1}|^{(p-1)/p} \text{ for } 3 \leq i \leq s_* - 1 \end{aligned}$$

Therefore for $3 \leq i \leq s_* - 1$

$$|A_{i+1}|^{p/(p-1)} \leq C(1 + h_0^{-1}) |Q^\theta_R(h_0)|^{\frac{1}{p-1}} |A_i - A_{i+1}|$$

Add up with respect to i from 3 to $s_* - 1$, we obtain

$$\begin{aligned} |A_{s_*}|^{\frac{p}{p-1}} &\leq \frac{C}{(s_* - 3) h_0^{\frac{1}{p-1}}} |Q^\theta_R(h_0)|^{\frac{1}{p-1}} |A_3 - A_{s_*}| \\ &\leq \frac{C}{(s_* - 3) h_0^{\frac{1}{p-1}}} |Q^\theta_R(h_0)|^{\frac{p}{p-1}} \end{aligned}$$

By choosing sufficiently large s_*, Lemma 3.2 easily follows.

For the case $p=q$, $1<p<2$, from Lemma 3.2 with $\omega=u$, we easily get

$$\text{meas}\left\{(x,t)\in Q_R^\theta(1):\ u(x,t)<\mu^-+\frac{\mu}{2^{s_*}}\right\}\le\alpha_1|Q_R^\theta(1)|$$

Notice that θ,η are independent on μ in this case. Similar to the proof of Lemma 3.1, if α_1 is sufficiently small, then

$$u(x,t)\ge\mu^-+\frac{\mu}{2^{s_*+1}},\quad \text{for } (x,t)\in K_R^\theta\times(-(1-h_0)R^p,0] \tag{3.10}$$

For the case $p<q$, $1<p<2$, the situation is more complicated. Rewrite

$$2\theta R=\left(\frac{\mu}{2^{s_*-2}}\right)^{\frac{p-q}{p}}2^{[(s_*-2)(p-q)+s^*(2-p)]/p}2R$$

Without loss of generality, we may enlarge s_* such that $2^{[(s_*-2)(p-q)+s^*(2-p)]+p}\ll h_0$ and $h_0/2^{[(s_*-2)(p-q)+s^*(2-p)]+p}$ is an integer. In $Q_R^\theta(h_0)$, construct the box of the following form

$$\hat{Q}_R=K_{2R}^\theta\times(\bar{t}-2^{[(s_*-2)(p-q)+s^*(2-p)]}(2R)^p,\bar{t}\,]$$

By Lemma 3.2, we can fine some $\hat{Q}_R\subset Q_R^R(h_0)$ such that

$$\text{meas}\left\{(x,t)\in\hat{Q}_R:\ \omega(x,t)<\mu^-+\frac{1}{4}\frac{\mu}{2^{s_*-2}}\right\}\le\alpha_1|\hat{Q}_R|$$

By Lemma 3.1 and its remark, it follows that for $0\le\mu^-<\dfrac{1}{8}\dfrac{\mu}{2^{s_*-2}}$

$$\omega(x,t)\ge\mu^-+\frac{1}{8}\frac{\mu}{2^{s_*-2}},\quad \text{for } (x,t)\in\hat{Q}_{\frac{R}{2}} \tag{3.11}$$

For $h\in[0,1-h_0]$, define

$$\mathcal{E}_h=\left\{t\in[-R^p,-(1-h_0)R^p]:u(x,t+hR^p)\ge\mu^-+\frac{\mu}{2^{s_*+1}},\ \text{for } x\in K_R^\theta\right\}$$

Then the set $\mathcal{E}_h$ is closed. If $0\le\mu'<\dfrac{\mu}{2^{s_*+1}}$, according to the above discussion, for any $h_j\in[0,1-h_0]$ $(1\le j\le m)$, by (3.11) we have

$$\mathcal{E}_{h_1}\cap\mathcal{E}_{h_2}\cap\cdots\cap\mathcal{E}_{h_m}\supset\left(\bar{t}-2^{[(s_*-2)(p-q)+s^*(2-p)]}\left(\frac{R}{2}\right)^p,\bar{t}\right]$$

Therefore, by the finite intersection property of compact set $[-R^p,-(1-h_0)R^p]$, we can find at least a point $t_*\in\bigcap\limits_h\mathcal{E}_h$. From the definition of $\mathcal{E}_h$, it follows that

$$u(x,t_*+hR^p)\ge\mu^-+\frac{\mu}{2^{s_*+1}},\ \text{for all } x\in K_R^\theta \tag{3.12}$$

Since for $0 < h_0 < \frac{1}{2}$, $[t_*, t_* + (1-h_0)R^p] \supset [-(1-h_0)R^p, -h_0R^p]$,

$$u(x,t) \geq \mu^- + \frac{\mu}{2^{s_*+1}}, \quad \text{for } (x,t) \in K_R^\theta \times [-(1-h_0)R^p, -h_0R^p] \tag{3.13}$$

From Lemma 3.1, Lemma 3.2, (3.10) and (3.13), we have

Theorem 3.1 *Assume that u is a supersolution of (1.1). If there exists some subbox $Q_{2R}^\eta(\bar{x}, 0) \subset Q_{2R}^\theta$ and α_0 dependent only on p, q, N, such that*

$$\text{meas}\left\{(x,t) \in Q_{2R}^\eta(\bar{x}, 0) : u(x,t) < \mu^- + \frac{\mu}{4}\right\} \leq \alpha_0 |Q_{2R}^\eta|$$

and for $h_0 \in \left(0, \frac{1}{2}\right)$, suppose that $0 \leq \mu^- + \frac{\mu}{2^{s_*+1}}$, where s_* depends only on p, q, N, s^*, h_0, then the following holds:

$$u(x,t) \geq \mu^- + \frac{\mu}{2^{s_*+1}}, \ \textit{for } (x,t) \in K_R^\theta \times [-(1-h_0)R^p, -h_0R^p]$$

4. Discussion in Case II

Throughout this section, we suppose that u is a subsolution of (1.1) and satisfies that for all subbox $Q_{2R}^\eta(\bar{x}, 0) \subset Q_{2R}^\theta$

$$\text{meas}\left\{(x,t) \in Q_{2R}^\eta(\bar{x}, 0) : u(x,t) > \mu^- + \frac{\mu}{4}\right\} \leq (1-\alpha_0)|Q_{2R}^\eta| \tag{4.1}$$

where α_0 is the same as that in (3.1).

We assume $0 \leq \mu^- < \frac{\mu}{2}$. From (4.1), we can find

$$\text{meas}\left\{(x,t) \in Q_{2R}^\eta(\bar{x}, 0) : u(x,t) > \frac{3\mu}{4}\right\} < (1-\alpha_0)|Q_{2R}^\eta| \tag{4.2}$$

for all $Q_{2R}^\eta(\bar{x}, 0) \subset Q_{2R}^\theta$.

Let $\theta_1 = 2^{s_1(2-p)/p}\mu^{(p-q)/p}$ and s_1 will be determined in Lemma 4.2 so that $s_1 < s^*$. Without loss of generality, we may enlarge s_1, s^*, such that $2^{s_1(2-p)/p}$ is an integer. Since $Q_{2R}^{\theta_1}(\bar{x}, 0)$ may be divided into finite disjoint box $Q_{2R}^\eta(\bar{\bar{x}}, 0)$, from (4.2), we obtain

$$\text{meas}\left\{(x,t) \in Q_{2R}^{\theta_1}(\bar{x}, 0) : u(x,t) > \frac{3\mu}{4}\right\} < (1-\alpha_0)|Q_{2R}^{\theta_1}| \tag{4.3}$$

where $Q_{2R}^{\theta_1}(\bar{x}, 0) \subset Q_{2R}^\theta$.

Lemma 4.1 *Assume (4.3) holds. The there exists $t^* \in \left[(-2R)^p, -\frac{\alpha_0}{2}(2R)^p\right]$, such that*

$$\text{meas}\left\{x \in K_{2R}^{\theta_1}(\bar{x}, 0) : u(x, t^*) > \frac{3\mu}{4}\right\} < \frac{1-\alpha_0}{1-\frac{\alpha_0}{2}}|K_{2R}^{\theta_1}| \tag{4.4}$$

For the proof of Lemma 4.1, refer to Lemma 5.1 of [4].

Lemma 4.2 *If* $0 \le \mu' < \frac{\mu}{2}$ *and* (4.4) *holds, then there exists* s_1 *dependent only on* p, q, N, *such that for all* $t \in \left[-\frac{\alpha_0}{2}(2R)^p, 0\right]$

$$\operatorname{meas}\{x \in K^{\theta_1}_{2R}(\bar{x}) : u(x,t) > (1-2^{-s_1})\mu\} < \Big(1 - \Big(\frac{\alpha_0}{2}\Big)^2\Big)|K^{\theta_1}_{2R}| \tag{4.5}$$

Proof In (2.6), take $H = \frac{\mu}{4}$, $k = \frac{3\mu}{4}$, $\nu = \frac{\mu}{2^{s_1}}$, then

$$\int_{K^{\theta}_{2R}(\bar{x})} \zeta^p(x) \Big[\int_{\frac{3\mu}{4}}^{\mu} \zeta^{q-2}(\psi^2)'\Big(\frac{\mu}{4}, \Big(s-\frac{3\mu}{4}\Big)^+, \frac{\mu}{2^{s_1}}\Big) ds\Big|_{t^*}^{t} dx$$
$$\le C\int_{t^*}^{t}\int_{K^{\theta_1}_{2R}(\bar{x})} |\nabla\zeta|^p \psi(\psi')^{2-p}\Big(\frac{\mu}{4}, \Big(u-\frac{3\mu}{4}\Big)^+, \frac{\mu}{2^{s_1}}\Big) dx dt$$

for $t \in [t^*, 0]$, $\zeta \in C^1_0\,(K^{\theta_1}_{2R}(\bar{x}))$, $\zeta \ge 0$. Taking cutoff function ζ such that $\zeta \equiv 1$ in $K^{\theta}_{(1-\sigma_1)2R}(\bar{x})$, $0 \le \zeta \le 1$ and $|\nabla\zeta| \le \frac{C}{\sigma_1\theta_1 R}$, we have

$$\Big[\int_{\frac{3\mu}{4}}^{(1-2^{-s_1})\mu} s^{q-2}(\psi^2)'\, ds\Big] \operatorname{meas}\{x \in K^{\theta_1}_{(1-\sigma_1)2R}(\bar{x}) : u(x,t) > (1-2^{-s_1})\mu\}$$
$$\le \int_{\frac{3\mu}{4}}^{\mu} s^{q-2}(\psi^2)' ds \operatorname{meas}\Big\{x \in K^{\theta_1}_{2R}(\bar{x}) : u(x,t) > \frac{3\mu}{4}\Big\}$$
$$+C\frac{1}{(\sigma_1\theta_1 R)^p}(\ln 2^{s_1-2})\Big(\frac{2^{s_1}}{\mu}\Big)^{2-p}(2R)^p|K^{\theta_1}_{2R}|$$
$$\le \Big[\int_{\frac{3\mu}{4}}^{\mu} s^{q-2}(\psi^2)' ds\Big]\frac{1-\alpha_0}{1-\frac{\alpha_0}{2}}|K^{\theta_1}_{2R}| + \frac{C(s_1-2)}{\sigma_1^p\mu^{2-q}}|K^{\theta_1}_{2R}|$$

Since

$$\Big[\int_{\frac{3\mu}{4}}^{(1-2^{-s})\mu} s^{q-2}(\psi^2)' ds \ge C_1(q)\mu^{q-2}\ln^2 2^{s_1-3}$$

and

$$\Big[\int_{\frac{3\mu}{4}}^{\mu} s^{q-2}\,(\psi^2)' ds\Big]\Big[\int_{\frac{3\mu}{4}}^{(1-2^{-s_1})\mu} s^{q-2}(\psi^2)' ds\Big]^{-1} - 1$$
$$\le \Big[C_2(q)\mu^{q-2}\int_{(1-2^{-s_1})\mu}^{\mu}(\psi^2)' ds\Big]\Big[C_1(q)\mu^{q-2}\int_{\frac{3\mu}{4}}^{(1-2^{-s_1})\mu}(\psi^2)' ds\Big]^{-1}$$
$$\le C\frac{2s_1}{(s_1-3)^2}$$

it follows that for all $t \in \Big[-\frac{\alpha_0}{2}(2R)^p, 0\Big]$,

$$\text{meas } \{x \in K_{2R}^{\theta_1}(\bar{x}) : u(x,t) > (1-2^{-s_1})\mu\}$$

$$\le \text{meas } \{x \in K_{2(1-\sigma_1)R}^{\theta_1}(\bar{x}) : u(x,t) > (1-2^{-s_1})\mu\} + N\sigma_1 |K_{2R}^{\theta_1}|$$

$$\le \Big\{\Big[1 + C\frac{2s_1}{(s_1-3)^2}\Big]\frac{1-\alpha_0}{1-\frac{\alpha_0}{2}} + C\frac{s_1-2}{\sigma_1^p(s_1-3)^2} + N\sigma_1\Big\}|K_{2R}^{\theta_1}|$$

It is easy to check that $\frac{1-\alpha_0}{1-\frac{\alpha_0}{2}} < 1 - \Big(\frac{\alpha_0}{2}\Big)^2$. Therefore, we can choose appropriate σ_1, s_1, so that (4.6) holds.

Let $Q_{2R}^{\theta_1}(\alpha_0) = K_{2R}^{\theta_1}(\bar{x}) \times \Big(-\frac{\alpha_0}{2}(2R)^p, 0\Big]$.

Lemma 4.3 *Suppose $0 \le \mu^- < \frac{\mu}{2}$ and (4.5) holds. Then for any $\alpha_2 > 0$, there exists some s^* dependent only on p, q, N, α_0, s_1, α_2 such that*

$$\text{meas } \{(x,t) \in Q_{2R}^{\theta_1}(\alpha_0) : u(x,t) > (1-2^{-s^*})\mu\} < \alpha_2 |Q_{2R}^{\theta_1}(\alpha_0)| \tag{4.6}$$

Proof Take $k_i = (1-2^{-i})\mu, i = s_1, s_1+1, \cdots, s^*$. By (4.5), apply Lemma 2.4 to the function $u, l = k_{i+1}$, $k = k_i$, we easily get

$$\Big(\frac{\mu}{2^{i+1}}\Big)\text{meas } A_{k_{i+1},\theta_1,2R}^+(t) \le \frac{C(2\theta_1 R)^{N+1}}{\Big(\frac{\alpha_0}{2}\Big)^2 |K_{2R}^{\theta_1}|}\int_{A_{k_i,\theta_1,2R}^+(t)\setminus A_{k_{i+1},\theta_1,2R}^+(t)} |\nabla u| dx \tag{4.7}$$

where $A_{l,\theta_1,2R}^+(t) = \{x \in K_{2R}^{\theta_1}(\bar{x}) : u(x,t) > l\}$, $t \in \Big[-\frac{\alpha_0}{2}(2R)^p, 0\Big]$. Integrate (4.7) with respect to t, denote

$$\bar{A}_i = \{(x,t) \in Q_{2R}^{\theta_1}(\alpha_0) : u(x,t) > k_i\}, i = s_1, \cdots, s^*$$

use Hölder's inequality, we get for all $s_1 \le i \le s^* - 1$

$$\frac{\mu}{2^{i+1}}|\bar{A}_{i+1}| \le C\theta_1 R\Big(\iint_{Q_{2R}^{\theta_1}(\alpha_0)} |\nabla(u-k_i)^+|^p dxdt\Big)^{\frac{1}{p}} |\bar{A}_i - \bar{A}_{i+1}|^{\frac{p-1}{p}} \tag{4.8}$$

By (2.2), (2.4), it is easy to get

$$\iint_{Q_{4R}^{\theta_1}(\alpha_0)} \zeta^p |\nabla(u-k_i)^+|^p dxdt$$

$$\le C\iint_{Q_{4R}^{\theta_1}(\alpha_0)} \{|\nabla\zeta|^p (u-k_i)^{+p} + \zeta^{p-1}|\zeta_t|\mu^{q-2}(u-k_i)^{+2}\}dxdt$$

Take a cutoff function ζ in $Q_{4R}^{\theta_1}(\alpha_0)$, such that $\zeta \equiv 1$ in $Q_{2R}^{\theta_1}(\alpha_0)$, $|\nabla\zeta| \le \frac{C}{\theta_1 R}$, $|\zeta_t| \le \frac{C}{R^p}$, then

$$\iint_{Q_{2R}^{\theta_1}(\alpha_0)} |\nabla(u-k_i)^+|^p dxdt \le C\Big\{\Big(\frac{1}{\theta_1 R}\Big)^p + \frac{\mu^{q-2}}{\alpha_0 R^p}\Big(\frac{\mu}{2^{s_1}}\Big)^{2-p}\Big\}\Big(\frac{\mu}{2^i}\Big)^p |Q_{4R}^{\theta_1}(\alpha_0)|$$

$$\le C\Big(\frac{1}{\theta_1 R}\Big)^p \Big(\frac{\mu}{2^i}\Big)^p |Q_{2R}^{\theta_1}(\alpha_0)| \tag{4.9}$$

using (4.8), (4.9), we conclude that for $s_1 \le i \le s^* - 1$

$$|\bar{A}_{i+1}|^{\frac{p}{p-1}} \le C|Q_{2R}^{\theta_1}(\alpha_0)|^{\frac{1}{p-1}}|\bar{A}_i - \bar{A}_{i+1}|$$

Add up with respect to i from s_1 to $s^* - 1$, observe that $|\bar{A}_i| \ge |\bar{A}_{s^*}|$ $(i \le s^*)$, we obtain $(s^* - s_1)|\bar{A}_{s^*}|^{p/(p-1)} \le C|Q_{2R}^{\theta_1}(\alpha_0)|^{p/(p-1)}$. By choosing s^*: $\dfrac{C}{s^* - s_1} \le \alpha_2^{p/(p-1)}$, (4.6) follows.

Remark s^* determines θ and the width of box Q_{4R}^θ.

Divide $Q_{2R}^\theta(\alpha_0) = K_{2R}^\theta \times \left[-\dfrac{\alpha_0}{2}(2R)^p, 0\right]$ into a finite number of subboxes $Q_{2R}^{\theta_1}(\alpha_0)$; for each $Q_{2R}^{\theta_1}(\alpha_0)$, (4.6) holds, therefore

$$\text{meas}\ \{(x,t) \in Q_{2R}^\theta(\alpha_0) : u(x,t) > (1 - 2^{-s^*})\mu\} < \alpha_2 |Q_{2R}^\theta(\alpha_0)| \tag{4.10}$$

Lemma 4.4 *Assume* $0 \le \mu^- < \dfrac{\mu}{2}$ *and* (4.10) *holds. If* α_2 *is small enough, then*

$$u(x,t) \le (1 - 2^{-(s^*+1)})\mu, \ \textit{for}\ (x,t) \in Q_R^\theta(\alpha_0) \tag{4.11}$$

Proof For $i = 0, 1, 2, \cdots$, set

$$k_i = [1 - 2^{-(s^*+1)} - 2^{-(s^*+1+i)}]\mu$$

$$R_i = R + \frac{R}{2^i}; \quad \bar{Q}_i = Q_{R_i}^\theta(\alpha_0)$$

By (2.2), it follows that

$$\sup_{-\frac{\alpha_0}{2}R_i^p < t < 0} \int_{K_{R_i}^\theta} \zeta^p \mu^{q-2}(u - k_{i+1})^{+2} dx + \iint_{\bar{Q}_i} \zeta^p |\nabla(u - k_{i+1})^+|^p dx dt$$
$$\le C \iint_{\bar{Q}_i} \{|\nabla\zeta|^p (u - k_{i+1})^{+p} + \zeta^{p-1}|\zeta_t|\mu^{q-2}(u - k_{i+1})^{+2}\} dx dt \tag{4.12}$$

Take ζ so that $\zeta \equiv 1$ in $\bar{Q}_{i+1}$, $|\nabla\zeta| \le \dfrac{C2^i}{\theta R}$, $|\zeta_t| \le \dfrac{C2^i}{\alpha_0 R^p}$, by Corollary 2.6, we have

$$\fint\!\!\fint_{Q_{i+1}} (u - k_{i+1})^{+p} dx dt \le C\frac{(\theta R_i)^p}{|\bar{Q}_i|}\left(\fint\!\!\fint_{Q_i} \chi[u > k_{i+1}] dx dt\right)^{\frac{p}{N+p}}$$
$$\cdot\Big[\sup_{-\frac{\alpha_0}{2}R_i^p < t \le 0} \frac{(\frac{\alpha_0}{2})}{\theta^p} \int_{K_{R_i}^{\theta_i}} \zeta^p (u - k_{i+1})^{+p} dx$$
$$+ \iint_{\bar{Q}_i} \left\{\zeta^p |\nabla(u - k_{i+1})|^{+p} + |\nabla\zeta|^p (u - k_{i+1})^{+p} dx dt\right\} dx dt\Big] \tag{4.13}$$

We estimate the first term in the bracket of the right side of (4.13):

$$\frac{1}{\theta^p}\int_{K_{R_i}^\theta} \zeta^p\,(u - k_{i+1})^{+p} dx \le 2^{(i+1)(2-p)}\mu^{q-2}\int_{K_{R_i}^\theta} \zeta^p (u - k_i)^{+2} dx$$
$$\le C2^i \iint_{\bar{Q}_i} \{|\nabla\zeta|^p (u - k_i)^{+p} + \zeta^{p-1}|\zeta_t|\mu^{q-2}(u - k_i)^{+2}\} dx dt \tag{4.14}$$

the last inequality is derived by (4.12) with k_i instead of k_{i+1}. Therefore by (4.12), (4.14), from (4.13) we get

$$\begin{aligned}
&\iint_{Q_{i+1}} (u-k_{i+1})^{+p} dx\,dt \\
&\leq C2^i \frac{(\theta R_i)^p}{|\bar{Q}_i|} \Big(\iint_{\bar{Q}_i} \frac{(u-k_i)^{+p}}{(k_{i+1}-k_i)^p} dx\,dt \Big)^{\frac{p}{N+p}} \\
&\cdot \iint_{\bar{Q}_i} \{ |\nabla\zeta|^p (u-k_i)^{+p} + \zeta^{p-1}|\zeta_t| \mu^{q-2}(u-k_i)^{+2} \} dx\,dt \\
&\leq C2^{3pi} (\theta R_i)^p \Big(\Big(\frac{\mu}{2^{s^*}}\Big)^{-p} \iint_{Q_i} (u-k_i)^{+p} dx\,dt \Big)^{\frac{p}{N+p}} \\
&\cdot \Big(\frac{1}{(\theta R)^p} + \frac{\mu^{q-2}}{\alpha_0 R^p} \Big(\frac{\mu}{2^{s^*}}\Big)^{2-p} \iint_{Q_i} (u-k_i)^{+p} dx\,dt \Big)
\end{aligned}$$

Since $R \leq R_i \leq 2R$, simple computation shows

$$\Big(\frac{\mu}{2^{s^*}}\Big)^{-p} \iint_{\bar{Q}_i} (u-k_{i+1})^{+p} dx\,dt \leq C2^{3pi} \Big[\Big(\frac{\mu}{2^{s^*}}\Big)^{-p} \iint_{\bar{Q}_i} (u-k_i)^{+p} dx\,dt \Big]^{1+p/(N+p)}$$

If α_2 in (4.10) is small enough, then $\Big(\frac{\mu}{2^{s^*}}\Big)^{-p} \iint_{Q_0} (u-k_0)^{+p} dx\,dt$ $(< \alpha_2)$ is small enough, according to Lemma 5.7 of [9], we have

$$u(x,t) \leq (1-2^{-(s^*+1)})\mu \quad \text{for } (x,t) \in Q_R^\theta(\alpha_0)$$

By Lemmas 4.1–4.4, we conclude

Theorem 4.1 *Suppose that u is a subsolution of* (1.1) *and that for all subbox $Q_{2R}^\eta(\bar{x},0) \subset Q_{2R}^\theta$*

$$\text{meas} \Big\{ (x,t) \in Q_{2R}^\eta(\bar{x},0) : u(x,t) > \mu^- + \frac{\mu}{4} \Big\} < (1-\alpha_0)|Q_{2R}^\eta|$$

and $0 \leq \mu^- < \frac{\mu}{2}$. Then

$$u(x,t) \leq (1-2^{-(S^*+1)})\mu, \quad \textit{for } (x,t) \in K_R^\theta \times \Big(-\frac{\alpha_0}{2}R^p, 0\Big]$$

5. The Proof of Theorem 1.1

To prove Theorem 1.1, choose h_0 such that $h_0 \leq \frac{\alpha_0}{16}$ and $1-h_0 \geq \frac{\alpha_0}{2}$. Clearly,

$$K_R^\theta \times \Big(-\frac{\alpha_0}{2}R^p, -\frac{\alpha_0}{16}R^p\Big] \subset \{K_R^\theta \times (-(1-h_0)R^p, -h_0R^p]\} \times \Big\{K_R^\theta \times \Big(-\frac{\alpha_0}{2}R^p, 0\Big]\Big\}$$

From Theorems 3.1 and 4.1, we have either $u(x,t) \geq \frac{\mu}{2^{s_*+1}}$ or $u(x,t) \leq (1-2^{-(s^*+1)})\mu$ for all $(x,t) \in K_R^\theta \times \Big(-\frac{\alpha_0}{2}R^p, -\frac{\alpha_0}{16}R^p\Big]$, where s^*, s_* depends only on p, q, N.

Suppose $u(x,t) \geq \frac{\mu}{2^{s_*+1}}$. Under transformations

$$\hat{\chi} = \frac{x}{\theta R}, \quad \hat{t} = \frac{t}{R^p}; \quad \hat{u}(\hat{x},\hat{t}) = u(\theta R\hat{x}, R^p\hat{t})/\mu$$

from Equation (1.1), $\hat{u}$ satisfies

$$(\hat{u}^{q-1})_t = 2^{-s^*(2-p)}\mathrm{div}_{\hat{x}}(|\nabla_{\hat{x}}\hat{u}|^{p-2}\nabla_{\hat{x}}\hat{u})$$

Set $\hat{v} = \hat{u}^{q-1}$, since $u \geq \mu/2^{s^*+1}$, we have

$$1 \geq \hat{v} \geq 2^{-(s_*+1)(q-1)}, \quad \text{in } K_1 \times \Big(-\frac{\alpha_0}{2}, -\frac{\alpha_0}{16}\Big]$$

Using Theorem 1 of [4], we have $[\hat{v}]_{\beta', K_{\frac{1}{2}} \times (-\frac{\alpha_0}{4}, -\frac{\alpha_0}{8}]} \leq C$, where $\beta' \in (0,1)$, C depends only on p, q, N. Therefore

$$|u(x_1,t_1) - u(x_2,t_2)| \leq C\mu\Big(\frac{\mu^{(q-p)/p}}{R}|x_1 - x_2| + \frac{1}{R}|t_1 - t_2|^{\frac{1}{p}}\Big)^{\beta_1} \tag{5.1}$$

for all $(x_1,t_1), (x_2,t_2) \in K^{\theta}_{\frac{R}{2}} \times \Big(-\frac{\alpha_0}{4}R^p, -\frac{\alpha_0}{8}R^p\Big]$, where β_1 depends on β', q.

Thus, we have either (5.1) holds or

$$u(x,t) \leq \sigma\mu, \quad \text{for } (x,t) \in K^{\theta}_{\frac{R}{2}} \times \Big(-\frac{\alpha_0}{4}R^p, -\frac{\alpha_0}{8}R^p\Big] \tag{5.2}$$

where $\sigma = 1 - 2^{-(s^*+1)}$.

Take $\mu = M$ and choose sufficiently small $R_0 = 4R$ such that $Q^{\theta}_{4R} \subset Q_T$. For any positive integer i, set

$$\begin{aligned}
&\mu_i = \sigma^i\mu_0, \quad \theta_i = 2^{s^*(2-p)/p}\mu_i^{(p-q)/p}, \ \mu_0 = \mu \\
&R_i = \bar{\sigma}R_{i-1}, \bar{\sigma} = \min\Big\{\frac{1}{8}\sigma^{(q-p)/p}, (\alpha_0 2^{-(3+2p)})^{\frac{1}{p}}\Big\} \\
&\bar{t}_i = \bar{t}_{i-1} - 2^{-(3+2p)}\alpha_0 R^p_{i-1}, \ \bar{t}_0 = 0 \\
&Q_{\mu_i}(R_i) = K^{\theta_i}_{R_i} \times (\bar{t}_i - R_i^p, \bar{t}_i]
\end{aligned} \tag{5.3}$$

It is easy to verify that $Q_{\mu_1}(R_1) \subset K^{\theta}_{\frac{R}{2}} \times \Big(-\frac{\alpha_0}{4}R^p, -\frac{\alpha_0}{8}R^p\Big)$. Hence, for all $(x,t) \in Q_{\mu_1}(R_1)$, either

$$\begin{aligned}
&|u(x_1,t_1) - u(x_2,t_2)| \\
&\quad \leq \frac{C}{R_1^{\beta_1}}(|x_1 - x_2| + |t_1 - t_2|^{\frac{1}{p}})^{\beta_1} \quad \text{for } (x_j,t_j) \in Q_{\mu_1}(R_1) \quad (j = 1,2)
\end{aligned}$$

or

$$u(x,t) \leq \mu_1 \quad \text{for } (x,t) \in Q_{\mu_1}(R_1)$$

By iterating this process, we conclude that either

$$u(x,t) \le \mu_i \quad \text{for all } (x,t) \in Q_{\mu_i}(R_i), i \ge 0 \tag{5.4}$$

or there exists some integer $l \ge 1$ so that

$$u(x,t) \le \mu_i \text{ for all } (x,t) \in Q_{\mu_i}(R_i),\ i = 1, \cdots, l-1 \tag{5.5}$$

$$\begin{aligned} &|u(x_1,t_1) - u(x_2,t_2)| \\ &\le \frac{C}{R_l^{\beta_1}}(|x_1 - x_2| + |t_1 - t_2|^{\frac{1}{p}})^{\beta_1}, \text{ for } (x_j,t_j) \in Q_{\mu_l}(R_l), \quad j = 1,2 \end{aligned} \tag{5.6}$$

In the first case, suppose (5.4) holds, then by (5.3)

$$\mu_i = \mu_0 \bar{\sigma}^{i \ln\sigma/\ln\bar{\sigma}} = \mu_0 \Big(\frac{R_i}{R_0}\Big)^{\ln\sigma/\ln\bar{\sigma}} = M_1 R_i^{\beta_2} \tag{5.7}$$

where $\beta_2 = \ln\sigma/\ln\bar{\sigma}$, $M_1 = \mu_0 R_0^{-\beta_2}$. Hence we have

$$0 \le u(x,t) \le M_1 R_i^{\beta_2} \text{ for all } (x,t) \in Q_{\mu_i}(R_i),\ i \ge 0 \tag{5.8}$$

Note that $\bar{t}_{i+1} \to -2^{-(3+2p)}\alpha_0(1-\bar{\sigma})^{-1}R_0^p \triangleq \tau_0$ as $i \to \infty$. Obviously, $(0,\tau_0) \in Q_{\mu_i}(R_i)(i \ge 0)$. From (5.8), we have

$$|u(x,t) - u(0,\tau_0)| \le 2M_1 R_1^{\beta_2} \quad \text{for } (x,t) \in Q_{\mu_i}(R_i)$$

For $(x,t) \in Q_{\mu_i}(R_i) \setminus Q_{\mu_{i+1}}(R_{i+1})$, $t \le \tau_0$, we also have

$$|u(x,t) - u(0,\tau_0)| \le 2M_1 R_1^{\beta_2} \tag{5.9}$$

and one of the following inequalities holds:

$$|x| \ge \theta_{i+1} R_{i+1} \quad \text{or} \quad t \le \bar{t}_{i+1} - R_{i+1}^p \tag{5.10}$$

By (5.3), (5.7), clearly

$$\theta_{i+1}R_{i+1} = 2^{s^*(2-p)/p}\sigma^{(p-q)/p}\bar{\sigma}M_1^{\frac{p-q}{p}}R_i^{[\frac{\beta_2(p-q)}{p}+1]} \tag{5.11}$$

Since $\beta_2 = \ln\sigma/\ln\bar{\sigma}$, $\sigma = 1 - 2^{-(s^*+1)}$, $\bar{\sigma} < \dfrac{1}{8}$, we may enlarge s^* so that β_2 is sufficiently small and $\beta_2\dfrac{q-p}{p} + 1 > 0$. By (5.10), we have

$$\begin{aligned} \tau_0 - t &= (\bar{t}_{i+1} - t) - (\bar{t}_{i+1} - \tau_0) \\ &\ge R_{i+1}^p + \Big[\sum_{j=0}^{i} 2^{-(3+2p)}\alpha_o R_j^p - \sum_{j=0}^{\infty} 2^{-(3+2p)}\alpha_i R_j^p\Big] \\ &= [1 - 2^{-(3+2p)}\alpha_0(1-\bar{\sigma}^p)^{-1}]\bar{\sigma}^p R_i^p \end{aligned} \tag{5.12}$$

For $(x,t)\in Q_{\mu_0}(R_0)=\bigcup_i Q_{\mu_i}(R_i)\setminus Q_{\mu_{i+1}}(R_{i+1})$, $t\le\tau_0$, there exists some i such that $(x,t)\in Q_{\mu_i}(R_i)\setminus Q_{\mu_{i+1}}(R_{i+1})$. Hence (5.9), (5.10) hold. Noting (5.11), (5.12), we have

$$\begin{aligned}|u(x,t)-u(0,\tau_0)| &\le 2M_1R_i^{\beta_2}\\ &\le 2M_1\max\{[(2^{s^*(2-p)/p}(\sigma M_1)^{(p-q)/p}\bar\sigma)^{-1}|x|]^{1/(\frac{p-q}{p}+\frac{1}{\beta_2})}\\ &\quad\cdot[(1-2^{-(3+2p)}\alpha_0(1-\bar\sigma(1-\bar\sigma^p)^{-1})^{-1}\bar\sigma^p(\tau_0-t)]^{\beta_2/p}\}\\ &\le M_2(|x|+|t-\tau_0|^{1/p})^{\beta_3}\end{aligned}\tag{5.13}$$

for $(x,t)\in Q_{\mu_0}(R_0)\bigcap\{t\le\tau_0\}=K_{R_0}^{\theta_0}\times(-R_0^p,\tau_0]$.

Now we turn to consider the second case, suppose that (5.5), (5.6) hold, similar to the proof of Lemma 6 of [7], it is easy to obtain

$$|u(x,t)-u(0,\tau_0)|\le C(|x|+|t-\tau_0|^{1/p})^{\beta_4},\ (x,t)\in K_{R_0}^{\theta_0}\times(-R_0^p,\tau_0]\tag{5.14}$$

where

$$\tau_0=-2^{-(3+2p)}\alpha_0(1-\bar\sigma^p)^{-1}R_0^p(>R_0^p)$$

By (5.13) (5.14), in a standard argument, Theorem 1.1 easily follows.

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