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ON THE CAUCHY PROBLEM FOR THE EQUATION OF FINITE-DEPTH FLUIDS*

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Abstract Some properties of the singular integral operator $G(\cdot)$ and the solvability of Cauchy problem for the singular integral-differential equations (1.1) and (1.2) of finite-depth fluids are studied.

Key Words Singular integral differential equation; Joseph equation; Cauchy problem; *a priori* estimates; Solvability.

Classification 35Q.

1. Introduction

In this paper we consider the Cauchy problem for the equation of finite-depth fluids [1]

$$U_t + 2UU_x - G(U_{xx}) = 0, \quad t \geq 0, x \in R \tag{1.1}$$

which was proposed by Joseph (1977) and later derived by Kubota and Dobbs (1978), where $U_t = \partial U/\partial t, U_x = \partial U/\partial x$ and etc. We also consider the generalized equation of finite-depth fluids with diffusion term

$$U_t = \alpha U_{xx} + \beta G(U_{xx}) + \varphi_x(U) \tag{1.2}$$

where $\varphi(\cdot)$ is assumed to be a mildly smooth function on R, such that

$$|\varphi^{(j)}(u)| \leq C(1 + |u|^{3-j}) \text{ for } j = 0, 1, \ u \in R \tag{A}$$

$$G(u) = P. \int_{-\infty}^{\infty} \frac{1}{2\delta}\Big(\coth\frac{\pi(x-y)}{2\delta} - \operatorname{sgn}(x-y)\Big)U(y)dy$$

is a singular integral operator; P. denotes the Cauchy principal value; α, β, δ are constants with $\alpha \geq 0, \delta > 0$. Equation (1.1) appears in the studying of oceanics and atmospheric science, which describes the evolution of long internal waves with small

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amplitude in a stably stratified fluid, propagating in one direction. The constant δ expresses the degree of depth. As the depth of fluid approaches to zero, equation (1.1) approximates to the well-known KdV equation [2]

$$U_t + 2UU_x + U_{xxx} = 0 \tag{1.3}$$

On the other hand, as δ tends to infinite, Equation (1.1) then reduces to the Benjamin-Ono equation

$$U_t + 2UU_x + H(U_{xx}) = 0 \tag{1.4}$$

where

$$H(U) = P.\int_{-\infty}^{\infty} \frac{U(y)}{\pi(y-x)} dy$$

is the Hilbert transform.

The KdV equation (1.3) and BO equation (1.4) have been studied extensively in their relation to the existence of a unique smooth solutions, the asymptotic behavior and soliton solution problems. According to the equation of finite-depth fluids (1.1) there are a few works [1–6] which concerned with the integrability and the solitary solutions, but the solvability of Cauchy problem has not been found to be discussed. In the present paper we shall concentrate on the Cauchy problem for the equation of finite-depth fluids (1.1), and the Cauchy problem for the generalized equation (1.2). Roughly speaking, we shall first study some properties of the singular integral operator $G(\cdot)$, then with the help of these properties we shall demonstrate the following results: in H^s with $s \geq 2$, the Cauchy problem for Equation (1.2) with $\alpha > 0$ is global well-posed and Equation (1.2) with $\alpha = 0$ is locally well-posed in a classical sense. For H^1, the Cauchy problem for the equation of finite-depth fluids (1.1) has at least one global solution in a weak class $L^\infty(0,T;H^1)$ for every initial data $U_0(x)$ given in H^1.

2. Preliminaries

We introduce the following notation: By $L^p(R), H^s(R)$ and $W_p^s(R)$ we denote the usual Sobolev spaces, the relative norms denoted respectively by $\|\cdot\|_p, \|\cdot\|_{H^s}$ and $\|\cdot\|_{W_p^s}$, where $p \geq 1$ is a real number, $s \geq 0$ is an integer number. By $W_p^{s,[\frac{s}{2}]}(Q_T)$ we denote the space of function $f(x,t)$ which has derivatives $D_t^r D_x^k f(x,t) \in L^p(Q_T)$ with $2r + k \leq s$, where $Q_T = R \times [0,T]$, T is an arbitrary positive number. By $W_{\infty,2}^{s,[\frac{s}{2}]}(Q_T)$ we denote the space of function $f(x,t)$ which has derivatives $D_t^r D_x^k f(x,t) \in L^\infty(0,T;L^2(R))$ with $2r + k \leq s$.

We define the Fourier transform $F[f]$ and inverse Fourier transform $F^{-1}[f]$ for function $f(x)$ as follows

$$F[f] = \int_{-\infty}^{\infty} f(x) e^{-2\pi i \xi x} dx, \quad F^{-1}[f] = \int_{-\infty}^{\infty} f(\xi) e^{2\pi i x \xi} d\xi$$

Lemma 2.1 *For any function $f(x)$ given in $L^2(R)$, we have*

(I) $\quad F[G(f)] = -i\Big(\coth 2\delta\pi\xi - \dfrac{1}{2\delta\pi\xi}\Big)F[f]$

(II) $\quad \|G(f)\|_2 \le \|f\|_2$

Proof By the definition of Fourier transform, we have

$$
\begin{aligned}
F[G(f)] &= \int_{-\infty}^{\infty} G(f)e^{-2\pi i\xi x}dx \\
&= -\frac{1}{2\delta}\int_{-\infty}^{\infty} e^{-2\pi i\xi x} P.\int_{-\infty}^{\infty}\Big(\mathrm{sgn}(x-y) - \coth\frac{\pi(x-y)}{2\delta}\Big)f(y)dydx \\
&= -\frac{1}{2\delta}\lim_{\varepsilon\to 0}\Big(\int_{-\infty}^{\infty}\int_{y+\varepsilon}^{\infty} e^{-2\pi i\xi x}\Big(\coth\frac{\pi(y-x)}{2\delta}+1\Big)f(y)dxdy \\
&\quad + \int_{-\infty}^{\infty}\int_{-\infty}^{y-\varepsilon} e^{-2\pi i\xi x}\Big(\coth\frac{\pi(y-x)}{2\delta}-1\Big)f(y)dxdy\Big) \\
&= -\frac{1}{2\delta}\lim_{\varepsilon\to 0}\Big(\int_{-\infty}^{\infty}\int_{\varepsilon}^{\infty} e^{-2\pi i\xi(z+w)}\Big(\coth\frac{-\pi z}{2\delta}+1\Big)f(w)dzdw \\
&\quad + \int_{-\infty}^{\infty}\int_{-\infty}^{-\varepsilon} e^{-2\pi i\xi(z+w)}\Big(\coth\frac{-\pi z}{2\delta}-1\Big)f(w)dzdw \\
&= -\frac{1}{2\delta}F[f]\lim_{\varepsilon\to 0}\Big(\int_{\varepsilon}^{\infty}(e^{2\pi i\xi z}-e^{-2\pi i\xi z})\Big(\coth\frac{\pi z}{2\delta}-1\Big)dz \\
&= -\frac{2i}{\delta}F[f]\lim_{\varepsilon\to 0}\Big(\int_{\varepsilon}^{\infty}\frac{\sin(2\pi\xi z)}{e^{\pi\lambda z}-1}dz \quad \Big(\lambda = \frac{1}{\delta}\Big) \\
&= -i\Big(\coth 2\delta\pi\xi - \frac{1}{2\delta\pi\xi}\Big)F[f]
\end{aligned}
$$

With the assertion of (I), by Parseval's identity we have

$$
\begin{aligned}
\int_{-\infty}^{\infty}(G(f))^2dx &= \int_{-\infty}^{\infty} F[G(f)\overline{F[G(f)]}d\xi \\
&= \int_{-\infty}^{\infty}\Big(\coth 2\delta\pi\xi - \frac{1}{2\delta\pi\xi}\Big)^2|F[f]|^2d\xi \\
&\le \int_{-\infty}^{\infty}|F[f]|^2d\xi = \int_{-\infty}^{\infty}(f(x))^2dx
\end{aligned}
$$

The proof of the lemma is now completed.

This lemma also shows that the transform $G(\cdot)$ is a linear bounded operator of Hilbert space $H^s(R)$ into itself for any $s \ge 0$.

Lemma 2.2 *Let $f(x), g(x)$ be given in $H^1(R)$. Then*

(I) $\quad G(f_x) = (G(f))_x$

(II) $\quad \displaystyle\int_{-\infty}^{\infty} fG(g)dx = -\int_{-\infty}^{\infty} gG(f)dx$

(III) $\quad (G(f_x))^2 - (f_x)^2 - 2G(f_xG(f_x))$

$$
= \frac{2}{\delta}[G(ff_x) - fG(f_x) + \int_{-\infty}^{x} f_xG(f_x)dx]
$$

Proof By the Fourier transform and Parseval identity, it is easy to prove (I) (II), which is omitted here. In what follows we shall demonstrate the equality (III).

In fact, let $G(f) = T(f) - L(f)$, here $L(f) = \frac{1}{2\delta} P. \int_{-\infty}^{\infty} \operatorname{sgn}(x-y) f(y) dy$. It is easy to show that

$$F[T(f)] = -i \coth(2\delta\pi\xi) F[f], \quad F(L(f)) = \frac{-i}{2\delta\pi\xi} F[f]$$

and

$$L(f_x) = (L(f))_x = \frac{1}{\delta} f(x) \tag{2.1}$$

In order to prove (III) we shall first check the following identity for the transform $T(\cdot)$

$$T(f_x)T(g_x) - f_x g_x = T(f_x T(g_x) + g_x T(f_x)) \tag{2.2}$$

Indeed, let $\varphi(\xi) = -i \coth(2\delta\pi\xi)$ for $\xi \neq 0$. We have

$$\begin{aligned} F[T(f_x)T(g_x) - f_x g_x] &= F[T(f_x)] * F[T(g_x)] - F[f_x] * F[g_x] \\ &= (\varphi(\xi) F[f_x]) * (\varphi(\xi) F[g_x]) - F[f_x] * F[g_x] \\ &= 4\pi^2 \int_{-\infty}^{\infty} (F[f](x))(F[g](\xi - x)) x(\xi - x)(1 - \varphi(x)\varphi(\xi - x)) dx \end{aligned}$$

On the other hand, we have

$$\begin{aligned} &F[T(f_x T(g_x) + g_x T(f_x))] = \varphi(\xi) F[f_x T(g_x) + g_x T(f_x)] \\ &= \varphi(\xi)(F[f_x] * F[T(g_x)] + F[g_x] * F[T(f_x)]) \\ &= 4\pi^2 \int_{-\infty}^{\infty} (F[f](x))(F[g](\xi - x)) x(\xi - x)(-\varphi(\xi)\varphi(\xi - x) - \varphi(x)\varphi(\xi)) dx \end{aligned}$$

We define

$$\varphi^*(x) = \begin{cases} x\varphi(x), & \text{as } x \neq 0 \\ -\dfrac{i}{2\delta\pi}, & \text{as } x = 0 \end{cases}$$

One can easily check that $\varphi^*(x)$ is a continuous function in R. In addition, by a simple computation, we get

$$x(\xi - x) - \varphi^*(x)\varphi^*(\xi - x) = -x\varphi(\xi)\varphi^*(\xi - x) - (\xi - x)\varphi(\xi)\varphi^*(x)$$

for any $x \in R$.

Therefore, on account of above identities, we obtain

$$F[T(f_x)T(g_x) - f_x g_x] = F[T(f_x T(g_x) + g_x T(f_x))]$$

which means the identity (2.2). Particularly, taking $f(x) = g(x)$, we then obtain

$$(T(f_x))^2 - (f_x)^2 = 2T(f_x T(f_x)) \tag{2.3}$$

With the above identity, we are going to prove the identity (III).

From (2.3), as $G(f) = T(f) - L(f)$, we have

$$(G(f_x) + L(f_x))^2 - (f_x)^2 = 2G(f_x(G+L)(f_x)) + 2L(f_x(G+L)(f_x))$$

By the properties (2.1) of the transform $L(\cdot)$, there is

$$\begin{aligned}(G(f_x))^2 - (f_x)^2 + \frac{1}{\delta^2}f^2 + \frac{2}{\delta}f(x)G(f_x) \\ = 2G(f_xG(f_x)) + \frac{2}{\delta}G(ff_x) + 2L(f_xG(f_x)) + \frac{2}{\delta}L(ff_x)\end{aligned} \tag{2.4}$$

In addition, we have

$$L(ff_x) = \frac{1}{2\delta}f^2$$

and

$$L(f_xG(f_x)) = \frac{1}{\delta}\int_{-\infty}^{x} f_xG(f_x)dx$$

With the above two identities, (2.4) achieves the result (III). The proof of the lemma is completed.

Corollary 2.1 *Let $f(x)$ be given in $W_3^1(R) \cap H^1(R)$. Then*

$$\int_{-\infty}^{\infty} f_x(G(f_x))^2dx + \frac{2}{\delta}\int_{-\infty}^{\infty} ff_xG(f_x)dx = \frac{1}{3}\int_{-\infty}^{\infty}(f_x)^3dx \tag{2.5}$$

Proof Multiplying the identity (III) in Lemma 2.2 by $f_x(x)$, integrating with respect to x in R, we then obtain the desired identity (2.5) by using the integration by parts.

Lemma 2.3 (Sobolev's inequality) *Let $f(x) \in L^q(R) \cap W_r^m(R)$ for $q, r \in [1, \infty]$. Then*

$$\|D_x^j f\|_p \le C\|D_x^m f\|_r^{\alpha}\|f\|_q^{1-\alpha} \tag{2.6}$$

where C is a constant independent of $f(x)$; j, m are nonnegative numbers that satisfy $j/m \le \alpha \le 1$ and $1/p = j + \alpha(1/r - m) + (1-\alpha)/q$.

Lemma 2.4 *Let $f(x) \in H^s(R), g \in H^{s-1}(R)$ and $h \in C^s(R)$, where s is a positive integer such that $s \ge 2$. Then*

$$\|D_x^s(fg) - fD_x^s g\|_2 \le C_s(\|D_x f\|_\infty \|D_x^{s-1}g\|_2 + \|g\|_\infty\|D_x^s f\|_2$$

and

$$\|D_x^s h(f)\|_2 \le C_s\sum_{n=1}^{s}(\|h^{(n)}(f)\|_\infty\|f\|_\infty^{n-1})\|D_x^s f\|_2$$

where the constant C_s depends only on s.

Proof The second calculus inequality is known, the reader is referred to [7] for proofs. Here we only verify the first one. Since

$$\|D_x^s(fg) - fD_x^s g\|_2 = \Big\|\sum_{j=0}^{s-1}\frac{s!}{j!(s-j)!}D_x^{s-j}fD_x^j g\Big\|_2$$

$$\leq C_s\sum_{j=0}^{s-1}\Big\|D_x^{s-j}f\Big\|_{\frac{2(s-1)}{s-1-j}}\Big\|D_x^j g\Big\|_{\frac{2(s-1)}{j}} \tag{2.7}$$

Applying the following Sobolev inequalities

$$\|D_x^{s-j}f\|_{\frac{2(s-1)}{s-1-j}} \leq C\|D_x^s f\|_2^{1-\alpha}\|D_x f\|_\infty^\alpha, \quad \|D_x^j g\|_{\frac{2(s-1)}{j}} \leq C\|D_x^{s-1}g\|_2^\alpha\|g\|_\infty^{1-\alpha}$$

in which $\alpha = j/(s-1), 0 \leq j \leq s-1$, we thus obtain from the identity (2.7)

$$\|D_x^s(fg) - fD_x^s g\|_2 \leq C_s\sum_{j=0}^{s-1}\|D_x^s f\|_2^{1-\alpha}\|D_x f\|_\infty^\alpha\|D_x^{s-1}g\|_2^\alpha\|g\|_\infty^{1-\alpha}$$

$$\leq C_s(\|D_x f\|_\infty\|D_x^{s-1}g\|_2 + \|g\|_\infty\|D_x^s f\|_2)$$

The proof is completed.

3. The Solvability for Equation (1.2) with $\alpha > 0$

In this section we shall prove the existence and uniqueness of smooth solution for the following Cauchy problem

$$U_t - \alpha U_{xx} - \beta G(U_{xx}) - \varphi_x(U) = 0 \tag{3.1}$$

$$U(x,0) = U_0(x) \tag{3.2}$$

where α, β are constants.

Lemma 3.1 *Let T be a positive number, $f(t)$ be a nonnegative function on $[0,T]$ such that*

$$\int_0^T f(t)dt \leq C_0, \quad f(t) \leq C_1 + C_2\int_0^t f^2(\tau)d\tau, \quad for\ t \in [0,T]$$

Then

$$\sup_{0\leq t\leq T} f(t) \leq C_1\exp(C_0C_2T)$$

Proof It is a direct consequence of Gronwall's inequality.

Lemma 3.2 *Let $b(x,t) \in W_\infty^{k,[k/2]}(Q_T), f(x,t) \in W_2^{k,[k/2]}(Q_T), U_0(x) \in H^{k+1}(R)$, and let α, β be any real numbers with $\alpha > 0$. Then the following Cauchy problem for a linear integral-differential equation, i.e.*

$$U_t - \alpha U_{xx} - \beta G(U_{xx}) + b(x,t)U_x = f(x,t) \tag{3.3}$$

(3.2) *has a unique global solution* $U(x,t) \in W_2^{k+2,[(k+2)/2]}(Q_T)$, *moreover,* $D_t^s D_x^{k+1-2s} U(x,t) \in L^\infty(0,T;L^2(R))$, *here* k, s *are integer numbers with* $0 \le s \le [(k+1)/2]$.

Proof By using the method of parametric extension, one could prove the results of the lemma exactly as what we did in [8]. The detailed process is omitted here.

We now extend the existence result of Cauchy problem for the linear equation (3.3) to the nonlinear equation (3.1) with $\varphi(\cdot)$ satisfing the condition (A) by using the fixed-point technique of Leray-Schauder and the method of induction.

Let α, β be constants with $\alpha > 0$, and let $U_0(x) \in H^1(R), \varphi(\cdot)$ satisfy condition (A).

We denote by B_0 the space $L^\infty(Q_T)$. On account of Lemma 3.2, we may define a mapping: $T_\lambda : B_0 \to B_0$ with a parameter $\lambda \in [0,1]$ as follows: For any $V \in B_0$, $U = T_\lambda V \in W_2^{2,1}(Q_t)$ be the solution of the following linear equation

$$U_t - \alpha U_{xx} - \beta G(U_{xx}) - \varphi'(V)U_x = 0$$

with the initial condition

$$U(x,0) = \lambda U_0(x) \tag{3.4}$$

Since the injection $W_2^{2,1}(Q_T) \to B_0$ is compact, it is easy to see that the mapping T_λ is completely continuous and $T_0(B_0) = 0$. Thus, in order to verify the existence of a generalized global solution of problem (3.1) (3.2), it remains to obtain a uniform boundedness in the space B_0 for all possible fixed-point of T_λ with $\lambda \in [0,1]$. Namely, we need to establish *a priori* estimate for the solution $U_\lambda(x,t)$ of the problem (3.1) (3.4).

Taking the product of Equation (3.1) by $U = U_\lambda(x,t)$ and integrating with respect to x in R. we get, by the integration by parts

$$\|U(\cdot,t)\|_2^2 + 2\alpha \int_0^t \|U_x(\cdot,t)\|_2^2 dt = \|U_0\|_2^2 \tag{3.5}$$

for any $t \in R$.

Moreover, we take the product of Equation (3.1) by U_{xx} and obtain

$$\frac{d}{dt}\int U_x^2 + 2\alpha \int U_{xx}^2 = 2\int \varphi_x(U)U_{xx} \le \alpha \int U_{xx}^2 + \frac{1}{\alpha}\int U_x^2 |\varphi'(U)|^2$$

Here we have

$$\frac{1}{\alpha}\int U_x^2|\varphi'(U)|^2 \le C(\alpha)\|\varphi'(U)\|_\infty^2 \|U_x\|_2^2 \le C(\alpha)(1 + \|U\|_\infty^4)\|U_x\|_2^2$$

$$\le C(\alpha)(1 + \|U\|_2^2\|U_x\|_2^2)\|U_x\|_2^2 \le C(\alpha, \|U_0\|_2)(1 + \|U_x\|_2^4)$$

where we have used the Sobolev inequality (2.6), condition (A) and the identity (3.5). We then obtain

$$\frac{d}{dt}\|U_x(\cdot,t)\|_2^2 + \alpha\|U_{xx}(\cdot,t)\|_2^2 \le C(\alpha, \|U_0\|_2)(1 + \|U_x\|_2^4)$$

Applying Lemma 3.1 with the identity (3.4), we get

$$\|U_x(\cdot,t)\|_2^2 + \alpha\int_0^t \|U_{xx}(\cdot,t)\|_2^2 dt \le C(\alpha, T, \|U_{0x}\|_2) \tag{3.6}$$

for any $t \in [0,T]$.

With the above inequalities (3.5), (3.6) and the Sobolev embedding theorem: $H^1(R) \to L^\infty(Q_T)$, we thus obtain the desired estimate

$$\|U\|_{B_0} \le C(\alpha, T, \|U_{0x}\|_2)$$

Theorem 3.1 *Let $U_0(x)$ be given in $H^1(R)$, and let α, β be real numbers with $\alpha > 0$, and $\varphi(\cdot)$ satisfy the condition* (A). *Then the Cauchy problem of the generalized equation with diffusion term, i.e.* (3.1) (3.2) *has a unique generalized solution $U(x,t) \in W_2^{2,1}(Q_T) \cap L^\infty(0,T;H^1(R))$.*

Proof The theorem is achieved as soon as we verify the uniqueness of the generalized solution $U(x,t)$.

Indeed, suppose that $U_1(x,t), U_2(x,t)$ are two solutions of the problem (3.1) (3.2). It is easy to see that the difference $W = U_1 - U_2$ such that

$$\begin{aligned} &W_t - \alpha W_{xx} - \beta G(W_{xx}) - (\varphi(U_1) - \varphi(U_2))_x = 0 \\ &W(x,0) = 0 \end{aligned} \tag{3.7}$$

Here we have

$$\begin{aligned}(\varphi(U_1) - \varphi(U_2))_x = &\frac{1}{2}(\varphi'(U_1) + \varphi'(U_2))W_x \\ &+ \frac{1}{2}(U_{1x} + U_{2x})\int_0^1 \varphi''(sU_1 + (1-s)U_2)ds W\end{aligned}$$

Therefore, by the fashion of estimates on the L^2-norms, we could easily prove that $W = U_1 - U_2 = 0$ for $(x,t) \in Q_T$.

The proof is completed.

Theorem 3.2 *Let $U_0(x)$ be given in $H^{k+1}(R), \varphi(\cdot)(\in C^{k+2})$ satisfy the condition* (A), *the constant $\alpha > 0$. Then the Cauchy problem of the generalized equation with diffusion term, i.e.* (3.1) (3.2) *has a unique global solution $U(x,t) \in W_2^{k+2,[k+2/2]}(Q_T) \cap W_{\infty,2}^{k+1,[k+1/2]}(Q_T)$, for $k \ge 0$.*

Proof We shall prove the theorem by induction.

For $k = 1$, as $U_0(x) \in H^2(R) \subset H^1(R)$ and $\varphi(\cdot) \in C^3(R) \subset C^2(R)$, by the result of Theorem 3.1, we conclude that the solution $U(x,t)$ of problem (3.1) (3.2) belongs to the space $W_2^{2,1}(Q_T) \cap L^\infty(0,T;H^1(R))$. Thus the Cauchy problem (3.1) (3.2) can be written as follows, with $U_x(x,t)$ replaced by $V(x,t)$

$$V_t - \alpha V_{xx} - \beta G(V_{xx}) = \varphi_{xx}(U), \quad V(x,0) = U_{0x}(x) \in H^1(R)$$

Since $U(x,t) \in W_2^{2,1}(Q_T) \cap L^\infty(0,T;H^1(R))$, together with the Sobolev inequality

$$\|U_x\|_4 \le C\|U_{xx}\|_2^{1/4}\|U_x\|_2^{3/4}$$

it is easy to check that $\varphi_{xx}(U) \in L^2(Q_T)$. Thus, by the results of Lemma 3.2, we see that $V \in W_2^{2,1}(Q_T) \cap L^\infty(0,T;H^1(R))$. This means that $U(x,t) \in W_2^{3,1}(Q_T) \cap L^\infty(0,T;H^2(R))$.

Moreover, suppose that the assertion of the theorem is true with $k = n \ge 1$, we have to justify that the solution $U(x,t)$ of the problem (3.1) (3.2) belongs to the space

$$W_2^{n+3,[(n+3)/2]}(Q_T) \cap W_{\infty,2}^{n+2,[(n+2)/2]}(Q_T) \quad \text{for } k = n+1$$

In fact, since $U_0(x) \in H^{n+2}(R) \subset H^{n+1}(R)$ and $\varphi(\cdot) \in C^{n+3} \subset C^{n+2}$, one can then find that the solution $U(x,t)$ of the problem (3.1) (3.2) belongs to $W_2^{n+2,[(n+2)/2]}(Q_T)$.

Differentiating Equation (3.1) $(n+1)$-times with respect to x, we obtain, with $D_x^{n+1}U(x,t)$ replaced by V

$$V_t - \alpha V_{xx} - \beta G(V_{xx}) = D_x^{n+2}\varphi(U), \quad V(x,0) = D_x^{n+1}U_0(x) \in H^1(R)$$

Notice that $U(x,t) \in W_2^{n+2,[(n+2)/2]}(Q_T)$ and the assertion of Lemma 2.4, one can easily see that $U_x(x,t) \in L^\infty(Q_T)$, moreover, $D_x^{n+2}\varphi(U) \in L^2(Q_T)$. With the help of Lemma 3.2, we conclude that

$$V = D_x^{n+1}U(x,t) \in W_2^{2,1}(Q_T) \cap L^\infty(0,T;H^1(R))$$

which, together with Equation (3.1), implies

$$U(x,t) \in W_2^{n+3,[(n+3)/2]}(Q_T) \cap W_{\infty,2}^{n+2,[(n+2)/2]}(Q_T)$$

The proof of the theorem is now completed.

4. Local Solvability for Equation (1.2) with $\alpha = 0$

This section is devoted to the local existence of a unique smooth solution for Equation (1.2) with $\alpha = 0$. On account of the results of Theorem 3.2, we have just to establish certain *a priori* estimates for the solution $U(x,t)$ of the problem (3.1) (3.2) with $\alpha > 0$ which are independent of α and which might be used to pass to the limit as α tends to zero.

We first state our result.

Theorem 4.1 *Let $U_0(x) \in H^s(R), s \ge 2$, and let $\varphi(\cdot)(\in C^{s+1}(R))$ satisfy the condition* (A) *and $|\varphi^{(3)}(u)| \le C_0(1+|u|^{s_0})$, here s_0 and s are positive integers, C_0 is a constant. Then the Cauchy problem* (1.2) (3.2) *with $\alpha = 0$ has a unique smooth solution $U(x,t)$ such that*

$$U(x,t) \in W_{\infty,2}^{s,[s/2]}(Q_{T_0})$$

where T_0 is a positive number which depends on s_0, s, C_0 and $\|U_{0zz}\|_2$.

Proof By a standard process, we can determine a finite interval $[0, T_0]$ in which we can establish some *a priori* estimates, then we obtain the local existence of smooth solution for the Cauchy problem (1.2) (3.2) with $\alpha = 0$.

Multiplying Equation (3.1) by $U = U_\alpha(x,t)$ (solution of the problem (3.1) (3.2) with $\alpha > 0$) and integrating over R, one can easily obtain

$$\sup_{0\le t\le T} \|U_\alpha(\cdot,t)\|_2 \le \|U_0\|_2 \tag{4.1}$$

In a similar manner, multiplying Equation (3.1) by $D_x^{2s}U$ and integrating by parts, we have

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int |D_x^s U|^2 + \alpha \int |D_x^{s+1}U|^2 &= (-1)^s \int D_x\varphi(U) D_x^{2s} U \\
&= \int D_x^s U\, D_x^s(\varphi'(U) D_x U) = \int D_x^s U [D_x^s(\varphi'(U)D_x U) - \varphi'(U) D_x^{s+1} U] \\
&\quad + \int \varphi'(U) D_x^s U\, D_x^{s+1} U \\
&\le \|D_x^s U\|_2 \|D_x^s(\varphi'(U)D_x U) - \varphi'(U) D_x^{s+1}U\|_2 \\
&\quad - \frac{1}{2}\int \varphi''(U) D_x U (D_x^s U)^2 \\
&\le C\|D_x^s U\|_2 (\|D_x\varphi'(U)\|_\infty \|D_x^s U\|_2 + \|D_x U\|_\infty \|D_x^s \varphi'(U)\|_2) \\
&\quad + \|\varphi''(U)\|_\infty \|D_x U\|_\infty \|D_x^s U\|_2^2 \\
&\le C(\|\varphi''(U)\|_\infty + \sum_{i=2}^{s+1} \|\varphi^{(i)}(U)\|_\infty) \|D_x U\|_\infty \|D_x^s U\|_2^2
\end{aligned} \tag{4.2}$$

where we have used the assertion of Lemma 2.4 and Hölder's inequality.

With the assumptions on $\varphi(\cdot)$, we have the following inequality from (4.2) for $s = 2$

$$\frac{d}{dt}\|D_x^2 U\|_2^2 + 2\alpha \|D_x^3 U\|_2^2 \le C(1 + \|U\|_\infty^{s'}) \|D_x U\|_\infty \|D_x^2 U\|_2^2$$

where $s' = \max\{s_0, 3\}$.

By the Sobolev inequalities

$$\|U\|_\infty \le C\|D_x^2 U\|_2^{\frac{1}{4}} \|U\|_2^{\frac{3}{4}}, \quad \|D_x U\|_\infty \le C\|D_x^2 U\|_2^{\frac{3}{4}} \|U\|_2^{\frac{1}{4}}$$

by a simple calculation, we yield

$$\frac{d}{dt}\|D_x^2 U\|_2^2 \le C(1 + \|D_x^2 U\|_2^{2l})$$

where $l = (11 + s')/8$, the constant C is independent of α.

With the help of Gronwall's inequality, it is easy to see that there is a constant $T_0 > 0$ which depends only on $l, \|D_x^2 U_0\|_2$ and the constant C, but α, such that

$$\sup_{0\le t\le T_0} \|D_x^2 U\|_2 \le C(T_0, s_0, \|D_x^2 U_0\|_2)$$

Therefore, by Sobolev's imbedding theorem, we see that $U \in L^\infty(Q_{T_0}), D_x U \in L^\infty(Q_{T_0})$. Once again, comming back to the inequality (4.2), we thus get, as $\alpha > 0$

$$\frac{d}{dt}\|D_x^s U\|_2^2 \le C(\|U\|_\infty, \|D_x U\|_\infty)\|D_x^s U\|_2^2$$

for $t \le T_0, s \ge 2$. This implies the following estimate

$$\sup_{0\le t\le T_0} \|D_x^s U\|_2 \le C(T_0, \|D_x^s U_0\|_2) \tag{4.3}$$

Combining (4.1) with (4.3), comming back to Equation (3.1), we finally obtain the desired estimate

$$\|U(x,t)\|_B \le C(T_0, s, \|U_0\|_{H^s})$$

where $B = W_{\infty,2}^{s,[s/2]}(Q_{T_0})$.

This means the existence of a local smooth solution of the Cauchy problem (1.2) (3.2). In order to complete the proof of the theorem, it remains to check the uniqueness of the solution.

Let U', U'' be two solutions that belong to the space $L^\infty(0,T;H^2(R))$. Accordingly, the difference function $W = U' - U''$ satisfies

$$W_t - \beta G(W_{xx}) - (\varphi_x(U') - \varphi_x(U'')) = 0, \quad W(x,0) = 0$$

Taking the L^2-inner product as usual, by a simple calculation, we may have

$$\|W(\cdot,t)\|_2^2 \le C(\|U'\|_{W_\infty^1}, \|U''\|_{W_\infty^1})\int_0^t \|W(\cdot,t)\|_2^2 dt$$

which implies that $W(x,t) = 0$ for $(x,t) \in Q_{T_0}$. This completed the proof of the theorem.

5. Global Solvability for Equation (1.1)

In this section we are going to verify the global existence of solution for the following Cauchy problem

$$U_t - G(U_{xx}) + 2UU_x = 0 \tag{5.1}$$

$$U(x,0) = U_0(x) \tag{5.2}$$

As a matter of fact, in this note we shall prove the existence of a weak solution for the problem (5.1) (5.2). In what follows, we shall first obtain certain *a priori* estimates, which are independent of α as α tends to zero, for the solution of the Cauchy problem (3.1) (3.2) with $\alpha > 0$, $\beta = 1$ and $\varphi(U) = U^2$, namely

$$U_t - \alpha U_{xx} - G(U_{xx}) + 2UU_x = 0, \quad U(x,0) = U_0(x) \tag{5.3}$$

Lemma 5.1 *Let $U = U_\alpha(x,t)$ be a smooth solution of the problem* (5.3) (5.2) *with* $\alpha > 0$. *Then*

$$\|U_\alpha(\cdot,t)\|_2 \leq \|U_0\|_2 \tag{5.4}$$

for $t \in [0,\infty)$.

Lemma 5.2 *Let $U = U_\alpha(x,t)$ be a smooth solution of the problem* (5.3) (5.2) *with* $\alpha > 0$. *Then*

$$\|U_x(\cdot,t)\|_2 \leq C \tag{5.5}$$

for $t \in [0,\infty)$, where C is a constant depends only on the norm $\|U_0\|_{H^1}$.

Proof By the assertion of Lemma 2.2, and the integration by parts, we find

$$(\mathrm{I}_1) \quad \frac{d}{dt}\int U_x^2 = 2\int U_x U_{xt} = -2\int U_{xx}(\alpha U_{xx} + G(U_{xx}) - 2UU_x)$$

$$= -2\alpha \int U_{xx}^2 - 2\int U_x^3$$

$$(\mathrm{I}_2) \quad \frac{d}{dt}\int (G(U_x))^2 = 2\int G(U_x)G(U_{xt})$$

$$= -2\int G(U_{xx})G(\alpha U_{xx} + G(U_{xx}) - 2UU_x)$$

$$= -2\alpha \int (G(U_{xx}))^2 + 4\int G(U_{xx})G(UU_x)$$

$$= -2\alpha \int (G(U_{xx}))^2 - 4\int UU_x G(G(U_{xx}))$$

$$(\mathrm{I}_3) \quad \frac{d}{dt}\int U^4 = 4\int U^3 U_t = 4\int U^3(\alpha U_{xx} + G(U_{xx}) - 2UU_x)$$

$$= 4\alpha \int U^3 U_{xx} - 12\int U^2 U_x G(U_x)$$

Combining the above three identities, we see that

$$\frac{d}{dt}\int \left(\frac{1}{2}U_x^2 + \frac{3}{2}(G(U_x))^2 + U^4\right) + \alpha \int U_{xx}^2 + 3\alpha \int (G(U_{xx}))^2$$
$$= 4\alpha \int U^3 U_{xx} - \int U_x^3 - 6\int UU_x G(G(U_{xx})) - 12\int U^2 U_x G(U_x) \tag{5.6}$$

In addition, we have

$$(\mathrm{I}_4)\qquad \frac{d}{dt}\int U^2G(U_x)=\int U^2G(U_{xt})+2\int UU_tG(U_x)$$
$$=-2\int UU_xG(\alpha U_{xx}+G(U_{xx})-2UU_x)$$
$$+2\int UG(U_x)(\alpha U_{xx}+G(U_{xx})-2UU_x)$$
$$=-2\alpha\int UU_xG(U_{xx})+2\alpha\int UU_{xx}G(U_x)-2\int UU_xG(G(U_{xx}))$$
$$-\int U_x(G(U_x))^2-4\int U^2U_xG(U_x)$$
$$(\mathrm{I}_5)\qquad \frac{d}{dt}\int UG(U_x)=2\int G(U_x)U_t$$
$$=2\int G(U_x)(\alpha U_{xx}+G(U_{xx})-2UU_x)$$
$$=2\alpha\int U_{xx}G(U_x)-4\int UU_xG(U_x)$$

Hence, from the last two identities, we get

$$\frac{d}{dt}\int(3U^2G(U_x)+\frac{3}{2\delta}UG(U_x))$$
$$=-6\alpha\int UU_xG(U_{xx})+6\alpha\int UU_{xx}G(U_x)+\frac{3\alpha}{\delta}\int U_{xx}G(U_x)$$
$$-6\int UU_xG(G(U_{xx}))-3\int U_x(G(U_x))^2-12\int U^2U_xG(U_x)$$
$$-\frac{6}{\delta}\int UU_xG(U_x) \tag{5.7}$$

Finally, on account of Eqs. (5.6) (5.7), we obtain

$$\frac{d}{dt}\int\Big[\frac{1}{2}U_x^2+\frac{3}{2}(G(U_x))^2+U^4-3U^2G(U_x)-\frac{3}{2\delta}UG(U_x)\Big]$$
$$+\alpha\int U_{xx}^2+3\alpha\int(G(U_{xx}))^2$$
$$=\alpha\Big[4\int U^3U_{xx}+6\int UU_xG(U_{xx})-6\int UU_{xx}G(U_x)-\frac{3}{\delta}\int U_{xx}G(U_x)\Big]$$
$$+3\Big[\int U_x(G(U_x))^2+\frac{2}{\delta}\int UU_xG(U_x)-\frac{1}{3}\int U_x^3\Big]$$
$$=\alpha\Big[4\int U^3U_{xx}+6\int UU_xG(U_{xx})-6\int UU_{xx}G(U_x)$$
$$-\frac{3}{\delta}\int U_{xx}G(U_x)\Big] \tag{5.8}$$

In the latter identity we have used the assertion of Corollary 2.1.

By Hölder's inequality and the following Sobolev's inequalities

$$\|U\|_6 \le C\|U_{xx}\|_2^{\frac{1}{6}}\|U\|_2^{\frac{5}{6}}, \quad \|U\|_\infty \le C\|U_{xx}\|_2^{\frac{1}{4}}\|U\|_2^{\frac{3}{4}}$$

$$\|U_x\|_2 \le \|U_{xx}\|_2^{\frac{1}{2}}\|U\|_2^{\frac{1}{2}}, \quad \|U_x\|_3 \le C\|U_{xx}\|_2^{\frac{7}{12}}\|U\|_2^{\frac{5}{12}}$$

the integrations in the right hand side of the identity (5.8) can be majorized as follows

$$\begin{aligned}
&\alpha\Big[4\int U^3U_{xx} + 6\int UU_xG(U_{xx}) - 6\int UU_{xx}G(U_x) - \frac{3}{\delta}\int U_{xx}G(U_x)\Big] \\
&\le C(\delta)\alpha[\|U_{xx}\|_2\|U\|_6^3 + \|U_x\|_3\|G(U_{xx})\|_2\|U\|_6 \\
&\quad + \|G(U_x)\|_2\|U\|_\infty\|U_{xx}\|_2 + \|U_{xx}\|_2\|U_x\|_2] \\
&\le C(\delta)\alpha[\|U_{xx}\|_2^{\frac{3}{2}}\|U\|_2^{\frac{5}{2}} + \|U_{xx}\|_2^{\frac{7}{4}}\|U\|_2^{\frac{5}{4}} + \|U_{xx}\|_2^{\frac{3}{2}}\|U\|_2^{\frac{1}{2}}] \\
&\le \frac{\alpha}{2}\|U_{xx}\|_2^2 + C(\delta, \|U_0\|_2)
\end{aligned}$$

where we have used Young's inequality and Lemma 2.1.

With the above inequality, we take the integration with respect to t for (5.8), by a simple calculation, we get

$$\begin{aligned}
&\frac{1}{2}\|U_x(\cdot,t)\|_2^2 + \frac{3}{2}\|G(U_x(\cdot,t))\|_2^2 + \|U(\cdot,t)\|_4^4 \\
&\le C(\delta, \|U_0\|_{H^1}) + 3\int U^2G(U_x) + \frac{3}{2\delta}\int UG(U_x) \\
&\le C(\delta, \|U_0\|_{H^1}) + 3\|G(U_x)\|_2\|U\|_4^2 + \frac{3}{2\delta}\|G(U_x)\|_2\|U\|_2 \\
&\le \|G(U_x)\|_2^2 + C(\delta, \|U_0\|_{H^1})(1 + \|U_x\|_2) \\
&\le \|G(U_x)\|_2^2 + \frac{1}{4}\|U_x\|_2^2 + C(\delta, \|U_0\|_{H^1})
\end{aligned}$$

where we have used Young's inequality and the fact that $\|U\|_4 \le 2^{\frac{1}{4}}\|U_x\|_2^{\frac{1}{4}}\|U\|_2^{\frac{3}{4}}$.

This completes the proof of the lemma.

Lemma 5.3 *Let $U = U_\alpha(x,t)$ be a smooth solution of the problem* (5.3) (5.2) *with $\alpha > 0$. Then*

$$\|U_t(\cdot,t)\|_{H^{-1}} \le C \tag{5.9}$$

for $t \in [0,\infty)$, where the constant C depends only on the norm $\|U_0\|_{H^1}$.

Proof For all $\psi(x,t) \in C_0^\infty(R)$, since $U(x,t) \in L^\infty(0,\infty; H^1)$, we have

$$\begin{aligned}
\int U_t\psi &= \int \psi(\alpha U_{xx} + G(U_{xx}) - 2UU_x) \\
&= -\alpha\int \psi_xU_x - \int \psi_xG(U_x) - 2\int \psi UU_x \\
&\le C(\|U_0\|_{H^1})\|\psi\|_{H^1}
\end{aligned}$$

where the the constant C is independent of α as α tends to zero. This achieves the result of the lemma.

With the above three lemmas, we now show that if for any $\alpha > 0, U = U_\alpha(x,t)$ is the solution of the Cauchy problem (5.3) (5.2) with $U_0(x)$ given in $H^1(R)$, then there exists a subsequence $U_{\alpha_i}(x,t)$ such that $U_{\alpha_i}(x,t)$ converges weakly star to a function $U(x,t)$ in $L^\infty(0,\infty;H^1(R))$, and the limit function $U(x,t)$ is a weak solution of (5.1) with initial data U_0. Namely we have

Theorem 5.1 *Let the initial data $U_0(x)$ be given in $H^1(R)$. Then the Cauchy problem of the equation of finite-depth fluids (5.1) (5.2) has at least one global weak solution $U = U(x,t)$, i.e.*

(I) $U(x,t) \in L^\infty(0,\infty;H^1(R))$

(II)
$$-\int_0^T \int U\psi_t dxdt + \int_0^T \int \psi_x G(U_x)dxdt - \int_0^T \int U^2 \psi_x dxdt = \int U_0(x)\psi(x,0)dx$$

for any number $T > 0$, where the test function $\psi(x,t) \in L^2(0,T;H_0^1(R))$ with $\psi_t \in L^2(0,T;L^2(R))$ and $\psi(x,T) = 0$.

Proof Let $U_\alpha = U_\alpha(x,t)$ be a smooth solution of problem (5.3) (5.2) with $\alpha > 0$. By the uniform estimates of Lemma 5.1, Lemma 5.2 and the weak compactness in a bounded reflexive Banach space, we can take subsequences if necessary, $U_\alpha \to U$ in the weak $*$-topology of $L^\infty(0,T;H^1(R))$, moreover, by the estimate of Lemma 5.3, together with Lemma 5.2 and Aubin's theorem [10], it follows that

$$U_\alpha(x,t) \to U(x,t) \text{ in } L^2(0,T;L^2(R)) \text{ strongly}$$

By a standard argument, it is easy to check that $U_\alpha U_{\alpha x} \to UU_x$ weakly in $L^2(0,T;L^2(R))$. Consequently, we immediately see that the limit function $U = U(x,t) \in L^\infty(0,\infty; H^1(R))$ is a solution of the Cauchy problem (5.1) (5.2) in the sense of distribution.

Corollary *Let $U = U(x,t) \in L^\infty(0,\infty;H^1)$ be a solution of problem (5.1) (5.2). Then*

$$\|U(x,t)\|_{C^{\left(\frac{1}{2},\frac{1}{4}\right)}(Q_\infty)} \le C$$

where C is a constant depending only on $\|U_0\|_{H^1}, Q_\infty = R^+ \times R$.

Proof Since $U(x,t) \in L^\infty(0,\infty;H^1)$, by Hölder's inequality, we have

$$|U(x_1,t) - U(x_2,t)| = \left| \int_{x_2}^{x_1} U_x(x,t)dx \right| \le |x_1 - x_2|^{\frac{1}{2}} \left(\int |U_x|^2 dx \right)^{\frac{1}{2}} \le C|x_1 - x_2|^{\frac{1}{2}}$$

or

$$\sup_{0 \le t < \infty} |U(x_1,t) - U(x_2,t)| \le C|x_1 - x_2|^{\frac{1}{2}} \tag{5.10}$$

for any $x_1, x_2 \in R$.

On the other hand, since $U_t \in L^\infty(0,\infty;H^{-1})$, then for all $\psi(x) \in L^2(R)$, setting $\psi^* = \int_0^x \psi(x)dx$, $V = \int_{-\infty}^x U dx$, we have

$$\begin{aligned}
\int \psi V_t &= \int \psi_x^* V_t = -\int \psi^* V_{xt} = -\int \psi^* U_t \\
&= -\int \psi^* [G(U_{xx}) - 2UU_x] = \int \psi [G(U_x) - U^2] \\
&\le C\|\psi\|_{L^2(R)}
\end{aligned}$$

This means that $V_t \in L^\infty(0,\infty;L^2(R))$. Thus by Sobolev inequality, we get

$$\begin{aligned}
\sup_{x\in R} |U(x,t_1) - U(x,t_2)| &= \|V_x(\cdot,t_1) - V_x(\cdot,t_2)\|_\infty \\
&\le C\|V(\cdot,t_1) - V(\cdot,t_2)\|_2^{\frac{1}{4}} \|V_{xx}(\cdot,t_1) - V_{xx}(\cdot,t_2)\|_2^{\frac{3}{4}} \\
&\le C|t_1 - t_2|^{\frac{1}{4}} \sup_t (\|V_t(\cdot,t)\|_2^{\frac{1}{4}} \|U_x(\cdot,t)\|_2^{\frac{3}{4}}) \\
&\le C|t_1 - t_2|^{\frac{1}{4}} \qquad (5.11)
\end{aligned}$$

On account of Eqs. (5.10), (5.11), the theorem is now proved.

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