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THE EXISTENCE OF TRAVELLING WAVE FRONT SOLUTIONS FOR REACTION-DIFFUSION SYSTEM*

Li Zhengyuan
(Institute of Maths., Peking University, Beijing 100871)
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Abstract In this paper by using upper-lower solution method the critical wave speed of wave front for a simplified mathematical model of Belousov-Zhabotinskii chemical reaction

$$u_t - u_{xx} = u(1 - u - rv)$$

$$v_t - v_{xx} = -buv$$

is obtained, where $0 < r < 1$, $b > 0$ are known.

Key Words Reaction-diffusion system; travelling wave front solutions; upper-lower solution method; B-Z reaction.
Classification 35K35

1. Introduction

In this paper we discuss the existence of travelling wave front solution of the following reaction-diffusion system

$$\begin{cases} u_t - u_{xx} = f(u,v) \\ v_t - v_{xx} = g(u,v) \end{cases} \tag{1}$$

A method for finding travelling wave front solution as limits of solutions of boundary value problems on finite domains has been developed by [1]-[4]. In the paper [4] by using the method and upper-lower solutions, a general principle for the existence of travelling wave front solution for the system (1) has been established. As an application of the general principle, necessary and sufficient condition for the existence of the monotone solution for the boundary value problem

$$\begin{cases} u'' + cu' + u(1 - r - u + rv) = 0 \\ v'' + cv' + bu(1 - v) = 0 \\ u(-\infty) = v(-\infty) = 0 \\ u(+\infty) = v(+\infty) = 1 \end{cases} \tag{2}$$

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was obtained: for $0 < r < 1$, $0 < b < (1-r)/r$, it was proved that (2) has a monotone solution if and only if $c \le -2\sqrt{1-r}$. In this paper we extend the above result for the existence of travelling wave front solutions to a principle which is much more convenient for practical use and in case of system (2) it does not require the limitation on b:

$$b < (1-r)/r$$

This also provides a similar result for the existence of monotone solution for (2).

2. Theorem of Existence

We first state our assumptions.

(H. 1): Let $l \ge 1$ and $a > 0$ be large enough.

(i) There exist a pair of upper and lower solutions $(\bar{u}_a, \bar{v}_a)$, $(\underline{u}_a, \underline{v}_a)$ for the following system on $[-a, a]$

$$\begin{cases} u'' + cu' + f(u,v) = 0 \\ v'' + cv' + g(u,v) = 0 \end{cases}$$

where $c \in \boldsymbol{R}$ is fixed and the functions f, g are as specified below. Moreover

$$0 < \underline{u}_a \le \bar{u}_a \le l, \quad 0 \le \underline{v}_a < \bar{v}_a \le l$$

(ii) $\bar{u}'_a(\xi) \ge 0$, $\underline{u}'_a(\xi) \ge 0$, $\bar{v}'_a(\xi) \ge 0$ for any $\xi \in [-a, a]$.

(iii) For any $\varepsilon > 0$ there exists $A > 0$ such that for any $\xi \in [-a, -A]$

$$\underline{v}_a(\xi) < \varepsilon, \quad \bar{v}_a(\xi) < \varepsilon$$

when $a > A$.

(H. 2):

(i) On $[0, l] \times [0, l]$, the only solutions of the equations

$$f(u,v) = 0, \quad g(u,v) = 0$$

are $(0, \alpha)(\alpha \in [0, l])$ and $(1, 1)$.

(ii) $f \in C^1([0,l] \times [0,l])$, $f(u,v) = uf_1(u,v)$, for $(u,v) \in (0,l) \times (0,l)$,

$$\frac{\partial f_1}{\partial u} < 0, \quad \frac{\partial f_1}{\partial v} > 0, \quad f_1(0,0) > 0$$

(iii) $g \in C^1([0,l] \times [0,l])$. There exists $l_0 \in (0, l]$ such that

$$g(u,v) > 0, \quad (u,v) \in (0,l) \times (0, l_0)$$

$$g(u,v) < 0, \quad (u,v) \in (0,l) \times (l_0, l), \quad \text{when} \quad l_0 < l$$

Our general principle is as follows.

Theorem 1 *Under assumptions* (H. 1) *and* (H. 2), *the system*

$$\begin{cases} u'' + cu' + f(u,v) = 0 \\ v'' + cv' + g(u,v) = 0 \\ u(-\infty) = v(-\infty) = 0 \\ u(+\infty) = v(+\infty) = 1 \end{cases} \tag{3}$$

has a monotone solution.

The proof of Theorem 1 is similar to the proof of Theorem 4.1 in the paper [4]. Thus we only outline the main idea of the proof.

1) Consider a BVP on finite interval $[-a, a]$ for $a > 0$ sufficiently large

$$\begin{cases} -u'' - cu' = f(u,v) \\ -v'' - cv' = g(u,v) \\ u(-a) = \underline{u}_a(-a), v(-a) = \underline{v}_a(-a) \\ u(a) = \bar{u}_a(a), v(a) = \bar{v}_a(a) \end{cases} \tag{4}$$

2) Using the upper and lower solution method we can prove that (4) has a solution $(u_a(\xi), v_a(\xi))$ and we get *a priori* estimates. Let m be any positive integer large enough, for $a \geq m$,

$$\|u_a\|_{C^{2+\gamma}[-m,m]} \leq M_m, \quad \|v_a\|_{C^{2+\gamma}[-m,m]} \leq M_m$$

where $\gamma \in (0,1)$ and M_m is independent of a. There exists a subsequence $a_n \to +\infty$ such that

$$\lim_{n\to\infty}(u_{a_n}(\xi), v_{a_n}(\xi)) = (u(\xi), v(\xi))$$

and $(u(\xi), v(\xi))$ satisfies the equation in (3).

3) We can prove that $(u_a(\xi), v_a(\xi))$ is monotone increasing in $\xi \in [-a, a]$ when $a > 0$ is sufficiently large.

4) From steps 2) and 3) we know that $(u(\xi), v(\xi))$ is increasing in $\xi \in (-\infty, +\infty)$. Obviously, the limits

$$\lim_{\xi\to+\infty}(u(\xi), v(\xi)) = (u_+, v_+), \quad \lim_{\xi\to-\infty}(u(\xi), v(\xi)) = (u_-, v_-)$$

exist. At last we can prove that

$$(u_+, v_+) = (1,1), \quad (u_-, v_-) = (0,0)$$

Thus $(u(\xi), v(\xi))$ is a monotone solution of (3).

3. Application

As an application of Theorem 1 we discuss a simplified mathematical model for Belousov-Zhabotinskii chemical reaction

$$\begin{cases} u_t - u_{xx} = u(1 - r - u + rv) \\ v_t - v_{xx} = bu(1 - v) \end{cases}$$

Now we can prove

Theorem 2 *Suppose that $0 < r < 1$, $b > 0$. Then there exists a number c^* with*

$$-2\sqrt{\max(1-r, br)} \leq c^* \leq -2\sqrt{1-r}$$

such that

(i) *If $c < c^*$, (2) has a monotone solution;*

(ii) *If $c > c^*$, (2) has no monotone solution;*

(iii) *If $b \leq (1-r)/r, c^* = -2\sqrt{1-r}$. If $b < (1-r)/r$ and $c = c^*$, (2) has a monotone solution.*

The proof of Theorem 2 is divided into the following lemmas.

Lemma 1 *Suppose that $0 < r < 1$, $b > 0$. Let $\beta \geq 1 - r$ and $\beta > br$. Then for any $c \leq -2\sqrt{\beta}$, (2) has a monotone solution.*

Proof It is easy to check that

$$f(u, v) = uf_1(u, v), \quad f_1(u, v) = 1 - r - u + rv$$

and

$$g(u, v) = bu(1 - v)$$

satisfy the assumption (H. 2) if we take $l_0 = 1$, $l > 1$. To prove Lemma 1 by using Theorem 1, we only need to find a pair of ordered upper and lower solutions of system

$$\begin{cases} -u'' - cu' = u(1 - r - u + rv) \\ -v'' - cv' = bu(1 - v) \end{cases} \tag{5}$$

on $(-a, a)$ satisfying (H. 1).

The following result is well-known. Let $\lambda > 0$ be a constant. Then the BVP

$$\begin{cases} -u'' - cu' = \lambda u(1 - u) \\ u(-\infty) = 0, \quad u(+\infty) = 1 \end{cases} \tag{6}$$

has a monotone solution if and only if $c \leq -2\sqrt{\lambda}$. For $\lambda = \beta$, $1 - r$ we denote the monotone solutions of (6) by $\hat{u}(\xi)$, $u_0(\xi)$, where $0 < r < 1$ is given and $\beta > 0$ is to be determined.

Let $\beta \geq 1-r$ and $\beta > br$. We can choose $k_1, k_2 > 1$ such that $(\bar{u}_a, \bar{v}_a) = (k_1\hat{u}(\xi), k_2\hat{u}(\xi))$ is the upper solution of (5). We can also choose $0 < l_a < 1$ such that $(\underline{u}_a, \underline{v}_a) = (l_a u_0, 0)$ is the lower solution of (5) with $(\underline{u}_a, \underline{v}_a) < (\bar{u}_a, \bar{v}_a)(\xi \in [-a, a])$. Now the conclusion of Lemma 1 comes from Theorem 1.

Lemma 2 *Suppose that* $0 < r < 1$, $b > 0$. *Then for any* $c > -2\sqrt{1-r}$, (2) *has no monotone solution.*

For proof see [3].

From Lemmas 1 and 2, we have

Lemma 3 *Suppose that* $0 < r < 1$, $b > 0$. *Set*

$$E = \{c | c \in R^1 \ \ such \ \ that (2) \ \ has \ \ monotone \ \ solution\}.$$

Then there exists the supremum of E

$$c^* = \sup E$$

with

$$-2\sqrt{\max(1-r, br)} \leq c^* \leq -2\sqrt{1-r}$$

Especially if $0 < b \leq (1-r)/r$, *then* $c^* = -2\sqrt{1-r}$.

Lemma 4 *Suppose that* $0 < r < 1$, $b > 0$.

(1) *For any* $c < c^*$, (2) *has a monotone solution;*

(2) *For any* $c > c^*$, (2) *has no monotone solution.*

Proof Let $c < c^*$. We consider the problem (2). There is $\bar{c} \in E$ with $c < \bar{c}$ and the problem

$$\begin{cases} -u'' - \bar{c}u' = u(1 - r - u + rv) \\ -v'' - \bar{c}v' = bu(1-v) \\ u(-\infty) = v(-\infty) = 0, \quad u(+\infty) = v(+\infty) = 1 \end{cases}$$

has a monotone solution, say $(\bar{u}(\xi), \bar{v}(\xi))$. Since $\bar{u}' > 0$, $\bar{v}' > 0$, it is clear that $(\bar{u}(\xi), \bar{v}(\xi))$ is an upper solution of (5). As in the proof of Lemma 1, we can choose $0 < l_a < 1$ such that $(\underline{u}_a, \underline{v}_a) = (l_a u_0, 0)$ is a lower solution of (5) with $(\underline{u}_a, \underline{v}_a) < (\bar{u}, \bar{v})$. Thus from Theorem 1 we see that (2) has a monotone solution for any $c < c^*$. According to the definition of c^*, the second claim is obviously true.

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References

[1] Berestycki, H., Nicolaenko, B. and Scheurer, B., Travelling wave solutions to combustion models and their singular limits, *SIAM J. Math. Anal.*, **16** (1985)., 1207–1242.

[2] Berestycki, H., Nicolaenko, B. and Scheurer, B., Wave solutions to reaction-diffusion systems modeling combustion, *Comtemp. Math.*, **77** (1983), 189–208.

[3] Li, Zhengyuan and Ye, Qixiao, Travelling wave front solutions for reaction-diffusion systems, J. Partial Differential Equations, **4** (3) (1991), 1–14.

[4] Ye, Qixiao and Wang, Mingxin. Travelling wave front solutions of Noyes-Field system for Belousov-Zhabotinskii reaction, *Nonlinear Analysis TMA*, **11** (1987), 1289–1302.