J. Partial Differential Equations
Vol. 3, No. 1(1990), 1—12

PERIODIC SOLUTIONS OF NONLINEAR WAVE EQUATIONS WITH DISSIPATIVE BOUNDARY CONDITIONS

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(Received Sept. 30, 1986; revised June 20, 1989)

Abstract Applying Nash-Moser's implicit function theorem, the author proves the existence of periodic solution to nonlinear wave equation

$$u_{tt} - u_{xx} + \varepsilon g(t, x, u, u_t, u_x, u_{tt}, u_{tx}, u_{xx}) = 0$$

with a dissipative boundary condition, provided ε is sufficiently small.

Key Words Nonlinear wave equation; time periodic solution; dissipative boundary condition.

Classifications 35L70; 35L20; 35B10.

0. Introduction

In this paper we discuss the existence of time-periodic solution for the following boundary value problem of nonlinear wave equation

$$F_\varepsilon(u) \equiv u_{tt} - u_{xx} + \varepsilon g(t, x, u, u_t, u_x, u_{tt}, u_{tx}, u_{xx}) = 0 \tag{0.1}$$

$$u(t, 0) = 0 \tag{0.2}$$

$$u_x(t, l) + \lambda u_t(t, l) = 0 \tag{0.3}$$

where g is periodic in t with period ω, $\lambda \neq 0$ is a constant. When (2) and (3) are Dirichlet's boundary conditions and there is a dissipative term λu_t in operator $F_\varepsilon(u)$, Rabinowitz in [1] proved the existence of periodic solution if ε is sufficiently small. The aim of this paper is to prove that if a dissipative boundary condition is given at one end of the interval instead of the term λu_t in the equation, then the problem admits a periodic solution provided ε is sufficiently small.

As the second order derivatives of u appear in the nonlinear term of the operator $F_\varepsilon(u)$, we shall use Nash-Moser's implicit function theorem to obtain the periodic solution of the problem. We shall apply the version of this theorem given by Moser in [2] which requires to solve the linearized equation only.

1. The Main Results

All functions mentioned in this section are periodic in t with the period ω. For simplicity, ω, l and λ are taken below to be 2π, 1 and 1, respectively.

Set

$$Q = [0, 2\pi] \times [0, 1]$$

and

$$U_p = \{u \mid \partial_t^j u \in H^5(Q), j \leqslant p, u(t, 0) = u_x(t, 1) + u_t(t, 1) = 0\},$$

$$F_p = \{u \mid \partial_t^j u \in H^s(Q), j \leqslant p\}$$

with norms

$$\| u \|_{U_p} = \max_{0 \leqslant j \leqslant p} \| \partial_t^j u \|_{H^s(Q)}$$

$$\| u \|_{F_p} = \max_{0 \leqslant j \leqslant p} \| \partial_t^j u \|_{H^s(Q)}$$

where $H^s(Q)$ are Sobolev's spaces on Q with norms $\| \cdot \|_{H^s(Q)}$. It is clear that U_p, F_p are Banach's spaces and

$$U_0 \supset U_1 \supset U_2 \supset \cdots, \quad F_0 \supset F_1 \supset F_2 \supset \cdots$$

For any $u \in U_p$, we can write it in the form of a Fourier's series

$$u = \frac{1}{2\pi} a_0(x) + \frac{1}{\pi} \sum_{j=1}^{\infty} (a_j(x) \cos jx + b_j(x) \sin jx)$$

We define the truncation operator T_N by

$$T_N u = \frac{1}{2\pi} a_0(x) + \frac{1}{\pi} \sum_{j \leqslant N} (a_j(x) \cos jx + b_j(x) \sin jx)$$

then it is easy to prove that

$$T_N: \quad U_r \longrightarrow U_{r+s}$$

and

$$\| T_N u \|_{U_{r+s}} \leqslant N^s \| u \|_{U_r} \tag{1.1}$$

$$\| (I - T_N) u \|_{U_r} \leqslant N^{-s} \| u \|_{U_{r+s}} \tag{1.2}$$

for any nonnegative integer N.

The linearized operator of the nonlinear operator $F_\varepsilon(u)$ is

$$F_\varepsilon'(u) v \equiv v_{tt} - v_{xx} + \varepsilon(a_{11} v_{tt} + a_{12} v_{tx} + a_{22} v_{xx} + a_1 v_t + a_2 v_x + a_0 v)$$

where $a_0 = g_u(t, x, u_{tt}, \cdots, u)$, $a_1 = g_{u_t}$, $a_2 = g_{u_x}$, $a_{11} = g_{u_{tt}}$, $a_{12} = g_{u_{tx}}$, $a_{22} = g_{u_{xx}}$.

In the following sections, we shall prove that if ε is sufficiently small, then $F_\varepsilon(u)$ satisfies the following for a constant b:

(1) If $u \in U_2$ and $\| u \|_{U_2} \leqslant b^{-1}$, then $F_\varepsilon(u) \in F_2$ and $F_\varepsilon'(u)$ is linear and bounded from U_2 into F_2.

(2) For any $u, v \in U_2$ and $\| u \|_{U_2} \leqslant b^{-1}$, $\| u + v \|_{U_2} \leqslant b^{-1}$, we have $\| F_\varepsilon(u+v) - F_\varepsilon(u) - F_\varepsilon'(u) v \|_{F_2} \leqslant b \| v \|_{U_2}^2$

(3) If $u \in U_{2+k}$ and $N \geqslant 1$ satisfy

$$\| u \|_{U_{2+\lambda}} \leqslant b^{-1} N^\lambda \quad \text{for} \quad \lambda = 0, 1, \cdots, k$$

then

$$\| F(u) \|_{F_{2+\lambda}} \leqslant b N^\lambda \quad \text{for} \quad \lambda = 0, 1, \cdots, k$$

(4) If $u \in U_{2+k}, h \in F_{2+k}$ and $N \geqslant 1$ satisfy

$$\| u \|_{U_{2+\lambda}} \leqslant b^{-1} N^\lambda, \quad \| h \|_{F_{2+\lambda}} \leqslant b N^\lambda \quad \text{for} \quad \lambda = 0, 1, \cdots, k$$

then the linearized equation

$$F_\varepsilon'(u) v = h \tag{1.3}$$

admits a solution $v \in U_k$ satisfying

$$\| v \|_{U_0} \leqslant b \| h \|_{F_2},$$

$$\| v \|_{U_\lambda} \leqslant b^2 N^\lambda \quad \text{for } (\lambda = 0, 1, \cdots, k$$

Once (1)—(4) are proved, we can obtain the main theorem immediately from Moser's theorem (see[2],[5]).

Main Theorem Suppose that g is sufficiently smooth, then exists $\varepsilon_0 \geqslant 0$ such that the boundary value problem (0.1)—(0.3) admits a solution $u \in U_2$ provided $\varepsilon \leqslant \varepsilon_0$.

Using the lemmas in Section 3, it is not difficult to prove (1), (2) and (3) above. So we shall devote our effort to prove (4) in rest sections of this paper.

2. Periodic Solution of Boundary Value Problem for Linear Equation

At first, we discuss the following boundary value problem for linear equation

$$L_\varepsilon u \equiv u_{tt} - u_{xx} + \varepsilon(a_{11}u_{tt} + a_{12}u_{tx} + a_{22}u_{xx} + a_1u_t + a_2u_x + a_0u) = h(t,x) \tag{2.1}$$

$$u(t,0) = 0, \quad u_x(t,1) + u_t(t,1) = 0 \tag{2.2}$$

$$u(t + 2\pi, x) = u(t,x) \tag{2.3}$$

where a are functions of t and x. Here we have shortened the notation writing a instead of $a_0, a_1, \cdots, a_{22}$.

In order to discuss the existence of periodic solution for problem (2.1)—(2.3), we consider the following Cauchy problem:

$$L_\varepsilon u = h, \quad \text{in} \quad Q_T = [0,T] \times [0,1] \tag{2.4}$$

$$u(t,0) = 0, \quad u_x(t,1) + u_t(t,1) = 0 \tag{2.5}$$

$$u(0,x) = \varphi_0(x), \quad u_t(0,x) = \varphi_1(x) \tag{2.6}$$

Theorem 2.1 *Let $a(t,x)$ be suitably smooth, $\varphi_0 \in H^{s+1}(0,1)$, $\varphi_1 \in H^s(0,1)$ and $h \in H^s(Q)$ satisfy compatibility conditions. Then Cauchy problem (2.4)—(2.6) admits a unique solution $u \in C^i([0,T]; H^{s+1-i}(0,1))$ $(i = 0, 1, \cdots, s+1)$ with boundary value $\partial_t^s u_t$ and $\partial_t^s u_x$ on boundary $x = 1$ in the following sense: there exists a sequence $\{u^{(k)}\}$ of smooth functions satisfying (2.5) such that*

$$\| u^{(k)} - u \|_{H^{s+1}(Q)} \to 0$$

$$\| L_\varepsilon u^{(k)} - h \|_{H^s(Q)} \to 0$$

$$\| u^{(k)}(0, \cdot) - \varphi_0 \|_{H^{s+1}(0,1)} + \| u_t^{(k)}(0, \cdot) - \varphi_1 \|_{H^s(0,1)} \to 0$$

$$\| \partial_t^s u_t^{(k)}(\cdot, 1) - \partial_t^s u_t(\cdot, 1) \|_{L^2(0,T)} + \| \partial_t^s u_x^{(k)}(\cdot, 1) - \partial_t^s u_x(\cdot, 1) \|_{L^2(0,T)} \to 0 \tag{2.7}$$

as $k \to \infty$, provided ε is sufficiently small.

Proof See [3].

Theorem 2.2 *Let $a \in F_{2+\lambda}$, $f \in F_{2+\lambda}$, $\varphi_0 \in H^{5+\lambda}(0,1)$ and $\varphi_1 \in H^{4+\lambda}(0,1)$, where λ is a nonnegative integer, and let the data satisfy compatibility conditions. Then Cauchy problem (2.4)—(2.6) admits a unique solution $u \in C^\lambda([0,T]; H^5(0,1))$ and u has the boundary values $\partial_t^{4+\lambda} u_t$ and $\partial_t^{4+\lambda} u_x$ at boundary $x = 1$ in the sense as in (2.7).*

Using the methods in [3], it is not difficult to prove this theorem.

Now, we discuss the periodic problem (2.1)—(2.3). Set

$$\widetilde{E}_\varepsilon(t)=\int_0^1[(1+a_{11}\varepsilon)u_t^2+(1-a_{22}\varepsilon)u_x^2]+\varepsilon u^2]dx$$

$$\widetilde{G}_\varepsilon(x)=\int_{\tau-\tau x}^{\tau+1+\tau x}[(1+a_{11}\varepsilon)u_t^2+(1-a_{22}\varepsilon)u_x^2+\varepsilon u^2]dt$$

In the following lemmas, we always assume that $\varepsilon>0$ is sufficiently small and C_1, C_2, $\cdots$ are positive constants. If u is a solution of Cauchy problem (2.4)—(2.6) with $h=0$, then we have the following estimates.

Lemma 2.1

$$\widetilde{E}_\varepsilon(t)\leqslant e^{C_1\varepsilon t}\widetilde{E}_\varepsilon(0)-(1-C_2\varepsilon)\int_0^t u_t^2(t,1)dt,\quad \forall\, t\in[0,T] \tag{2.8}$$

Proof Without loss of generality, we can assume u is sufficiently smooth. It is easy to verify

$$\begin{aligned}\frac{d\widetilde{E}_\varepsilon(t)}{dt}&=2\varepsilon\int_0^1[a_1u_t+a_2u_x+a_0u]u_tdx\\&+\varepsilon\int_0^1[\partial_ta_{11}u_t^2-\frac{1}{2}\partial_xa_{12}u_t^2-\partial_ta_{22}u_x^2-\partial_xa_{22}u_xu_t+2uu_t]dx\\&-(1-a_{12}\varepsilon-a_{22}\varepsilon)u_t^2(1,t)\end{aligned} \tag{2.9}$$

Taking into account

$$\int_0^1u^2dx=\int_0^1\Big(\int_0^x u_x(\xi,t)d\xi\Big)^2dx\leqslant\int_0^1u_x^2dx$$

from (2.9) we have

$$\frac{d\widetilde{E}_\varepsilon(t)}{dt}\leqslant-(1-C_2\varepsilon)u_t^2(t,1)+C_1\varepsilon\widetilde{E}_\varepsilon(t) \tag{2.10}$$

The lemma is obtained immediately from (2.10).

Lemma 2.2

$$\widetilde{E}_\varepsilon(t)\leqslant(C_3\varepsilon+e^{C_1\varepsilon t})\widetilde{E}_\varepsilon(0)-C_4\widetilde{G}_\varepsilon(1),\quad\forall\, t\in[0,T], \tag{2.11}$$

Proof Using

$$u(t,1)=u(0,1)+\int_0^t u_t(\eta,1)d\eta$$

it is not difficult to estimate

$$\int_0^t u^2(t,1)dt\leqslant C\int_0^1u_x^2(0,x)dx+C\int_0^t u_t^2(t,1)dt \tag{2.12}$$

Noting the boundary condition (2.5) at $x=1$, from (2.8) and (2.12), we have

$$\begin{aligned}\widetilde{E}_\varepsilon(t)\leqslant&e^{C_1\varepsilon t}\widetilde{E}_\varepsilon(0)\\&-\frac{1}{2}(1-C_2\varepsilon)\int_0^t(u_t^2+u_x^2+\varepsilon u^2)(t,1)dt\\&+\frac{1}{2}(1-C_2\varepsilon)C\varepsilon\int_0^tu_t^2(t,1)dt+\frac{1}{2}(1-C_2\varepsilon)C\varepsilon\int_0^1u_x^2(0,x)dx\end{aligned}$$

The lemma is obtained easily from above estimate.

Lemma 2.3 *If $\tau>1$, then*

$$\widetilde{G}_\varepsilon(x)\leqslant C_5\widetilde{G}_\varepsilon(1),\quad\forall\, x\in[0,1] \tag{2.13}$$

Proof Without loss of generality, we can assume u is sufficiently smooth. Then we have

$$\frac{d}{dx}\tilde{G}_\varepsilon(x) = \tau[(1+a_{11}\varepsilon)u_t^2 + (1-a_{22}\varepsilon)u_x^2 + \varepsilon u^2](x,\tau+1+\tau x)$$
$$+ \tau[(1+a_{11}\varepsilon)u_t^2 + (1-a_{22}\varepsilon)u_x^2 + \varepsilon u^2](x,\tau-\tau x)$$
$$+ 2\int_{\tau-\tau x}^{\tau+1+\tau x}[(1+a_{11}\varepsilon)u_t u_{xt} + (1-a_{22}\varepsilon)u_x u_{xx} + \varepsilon u u_x$$
$$+ \varepsilon\partial_x a_{11}u_t^2 - \varepsilon\partial_x a_{22}u_x^2](x,t)dt \tag{2.14}$$

Using the formula for integration by parts if it need be, from (2.14) we can obtain

$$\frac{d}{dx}\tilde{G}_\varepsilon(x) = \tau[(1+a_{11}\varepsilon)u_t^2 + (1-a_{22}\varepsilon)u_x^2 + \varepsilon u^2](x,\tau+1+\tau x)$$
$$+ \tau[(1+a_{11}\varepsilon)u_t^2 + (1-a_{22}\varepsilon)u_x^2 + \varepsilon u^2](x,\tau-\tau x)$$
$$+ 2(1+a_{11}\varepsilon)u_t u_x(x,\tau+1+\tau x) - 2(1+a_{11}\varepsilon)u_t u_x(x,\tau-\tau x)$$
$$+ a_{12}\varepsilon u_x^2(x,\tau+1+\tau x) - a_{12}\varepsilon u_x^2(x,\tau-\tau x)$$
$$+ \varepsilon\int_{\tau-\tau x}^{\tau+1+\tau x}[2u_x(a_1u_t + a_2u_x + a_0u) - 2\partial_t a_{11}u_t u_x$$
$$+ 2uu_x + \partial_x a_{11}u_t^2 - \partial_x a_{22}u_x^2 - \partial_t a_{12}u_x^2](x,t)dt \tag{2.15}$$

Under the hypothesis $\tau > 1$, if ε is sufficiently small, from (2.15) we have

$$\frac{d}{dx}\tilde{G}_\varepsilon(x) \geqslant -C\tilde{G}_\varepsilon(x) \tag{2.16}$$

The lemma follows from (2.16).

Lemma 2.4 *Let $T > 1$, then*

$$\tilde{E}_\varepsilon(T) \leqslant q\tilde{E}_\varepsilon(0) \tag{2.17}$$

where $0 < q < 1$.

Proof From Lemma 2.2 we have

$$\tilde{E}_\varepsilon(T) \leqslant (C_3\varepsilon + e^{C_1\varepsilon T})\tilde{E}_\varepsilon(t), \quad \forall\, t \in [0,T]$$

From the above and Lemma 2.3, we can obtain

$$\tilde{E}_\varepsilon(T) \leqslant (C_3\varepsilon + e^{C_1\varepsilon T})\int_\tau^{\tau+1}\tilde{E}_\varepsilon(t)dt$$
$$\leqslant (C_3\varepsilon + e^{C_1\varepsilon T})\int_{\tau-\tau x}^{\tau+1+\tau x}\tilde{E}_\varepsilon(t)dt$$
$$= (C_3\varepsilon + e^{C_1\varepsilon T})\int_0^1 G_\varepsilon(x)dx$$
$$\leqslant C_5(C_3\varepsilon + e^{C_1\varepsilon T})G_\varepsilon(1) \tag{2.18}$$

From (2.18) and Lemma 2.2, it follows that

$$\tilde{E}_\varepsilon(T) \leqslant (C_3\varepsilon + e^{C_1\varepsilon T})\tilde{E}_\varepsilon(0) - \frac{C_4}{C_5(C_3\varepsilon + e^{C_1\varepsilon T})}\tilde{E}_\varepsilon(T)$$

If ε is sufficiently small, the above gives (2.17). The lemma is proved.

Lemma 2.5 *Under the hypothesis $T > 1$, then*

$$E(T) \leqslant qE(0) \tag{2.19}$$

with $0 < q < 1$, *provided ε is sufficiently small, here* $E(t) = \tilde{E}_0(t)$

Proof Noting

$$\int_0^1 u^2(t,x)dx \leqslant \int_0^1 u_x^2(t,x)dx, \quad \forall\, t \in [0,T]$$

(2. 19) follows from Lemma 2. 4.

Theorem 2. 3 *Let h and the coefficients a of L, be periodic functions in t with period 2π, and let $h \in L^2(Q)$, $a_{ij} \in C^1(\bar{Q})$, a_j and $a_0 \in C^0(\bar{Q})$. Then the boundary value problem (2. 1)—(2. 3) admits a periodic solution $u \in C^{1-i}([0,2\pi];H^i(0,1))(i=0,1)$ in the sense of Theorem 2. 1.*

Proof Set

$$H = \dot{H}^1(0,1) \times L^2(0,1)$$

where $\dot{H}^1(0,1) = \{u \mid u \in H^1(0,1) \text{ and } u(0) = 0\}$. For $\varphi = (\varphi_0,\varphi_1)$, the norm is defined as

$$\| \varphi \|_H^2 = \| \partial_x \varphi_0 \|_{L^2(0,1)}^2 + \| \varphi_1 \|_{L^2(0,1)}^2$$

If we take $\varphi \in H$ as the data in (2. 6), from Theorem 2. 1 it follows that Cauchy problem (2. 4)—(2. 6) admits a solution $u \in C^{1-i}([0,T];H^i(0,1))(i=0,1)$ for any $T>0$. Let Φ be a map

$$\Phi: \quad (\varphi_0,\varphi_1) \longrightarrow (u(2\pi,\cdot),u_t(2\pi,\cdot))$$

where u is the solution of Cauchy problem with data φ_0 and φ_1. It is clear that Φ maps H into H.

If $2m\pi > 1$, from Lemma 2. 5 we have

$$\| \Phi^m \varphi - \Phi^m \psi \|_H^2 \leqslant q \| \varphi - \psi \|_H^2$$

Therefore, Φ^m has a fixed point $\bar{\varphi} \in H$. It is easy to verify that $\bar{\varphi}$ is also a fixed point of map Φ. So the solution of Cauchy problem with initial data $\bar{\varphi} = (\bar{\varphi}_0,\bar{\varphi}_1)$ is a periodic solution. The statement of Theorem 2. 3 is proved.

Theorem 2. 4 *Let a and h in F_{2+k} be periodic functions in t with period 2π, where $k \geqslant 0$ is an integer. Then problem (2. 1)—(2. 3) admits a periodic solution*

$$u \in C^k([0,2\pi]; \quad H^5(0,1))$$

Proof Set

$$B = H^1(0,1) \times \cdots \times H^1(0,1)$$

Let $\varphi_0 \in H^{5+k}(0,1)$ and $\varphi_1 \in H^{4+k}(0,1)$ be the initial data at $t=0$ which satisfy compatibility conditions. Then from the equation (2. 4) and the data we can determine uniquely

$$\varphi^{(0)} = (u(0,\cdot),\partial_t u(0,\cdot),\cdots,\partial_t^{4+k} u(0,\cdot)) \in B$$

Let u be the solution of Cauchy problem (2. 4)—(2. 6) and $m>0$ be an integer satisfying $\tilde{\omega} = 2m\pi > 1$. For any integer $l>0$, set

$$\varphi^{(l)} = [u(l\tilde{\omega},\cdot),\partial_t u(l\tilde{\omega},\cdot),\cdots,\partial_t^{4+k} u(l\tilde{\omega},\cdot)]$$

It is clear that $\varphi^{(l)} \in B$. We denote

$$\varphi^{(l+1)} = \Phi \varphi^{(l)}$$

As in the proof of Theorem 2. 3, we can prove that Φ is contract in B. Therefore, there exists $\varphi^{(\infty)} \in B$ such that

$$\varphi^{(l)} \longrightarrow \varphi^{(\infty)} \quad \text{in } B, \quad \text{as } l \to \infty$$

We extend continuously Φ to $\varphi^{(\infty)}$, then $\Phi\varphi^{(\infty)} = \varphi^{(\infty)}$. Set

$$\varphi^{(\infty)} = (\varphi_0^{(\infty)},\varphi_1^{(\infty)},\cdots,\varphi_{4+k}^{(\infty)})$$

Then the solution $u^{(\infty)}$ of Cauchy problem with initial data $\varphi_0^{(\infty)}$ and $\varphi_1^{(\infty)}$ satisfies

$$u^{(\infty)}(\tilde{\omega},x)=u^{(\infty)}(0,x)=\varphi_0^{(\infty)}$$
$$u_t^{(\infty)}(\tilde{\omega},x)=u_t^{(\infty)}(0,x)=\varphi_1^{(\infty)}$$
That is, $u^{(\infty)}$ is a periodic solution in t with period $\tilde{\omega}$. If we write
$$u^{(l)}(t,x)=u(l\tilde{\omega}+t,x),\quad \forall\, t\in[0,\tilde{\omega}]$$
then
$$u^{(l)}\in C^{k+5-i}([0,\tilde{\omega}];\quad H^i(0,1)),\quad i=0,1$$

It is not difficult to see that if we treat $u^{(\infty)}$ as a function defined in $[0,\tilde{\omega}]\times[0,1]$, then it is the limit of $\{u^{(l)}\}$ in $C^{k+5-i}([0,\tilde{\omega}];H^i(0,1))\,(i=0,1)$. Therefore, $u^{(\infty)}\in C^{k+5-i}([0,\tilde{\omega}];H^i(0,1))\,(i=0,1)$. From Theorem 2.3, we know it is periodic in t with period 2π. Theorem 2.4 is proved.

3. Solution of Linearized Equation and Its Estimates

In order to prove (4) in Section 1, we need the following lemmas.

Lemma 3.1 *Let $m\geqslant 2$ and $w_1,\cdots,w_l\in H^m(Q)$,*
$$\sum_{j=1}^{l}|\alpha_j|\leqslant m$$
then
$$\|\partial^{\alpha_1}w_1\partial^{\alpha_2}w_2\cdot\cdots\cdot\partial^{\alpha_l}w_l\|_{L^2(Q)}\leqslant C\prod_{j=1}^{l}\|w_j\|_{H^m(Q)}$$

Proof See [4].

Lemma 3.2 *Let $g(t,x,w_1,\cdots,w_6)$ be sufficiently smooth, and $w_j\in F_{2+\lambda}$, $\lambda=0,1,\cdots,k$. Then $g\in F_{2+\lambda}$. Moreover, if*
$$\|w_j\|_{F_{2+\lambda}}\leqslant b_1N^\lambda,\quad j=1,\cdots,6,\quad \lambda=0,1,\cdots,k$$
then
$$\|g\|_{F_{2+\lambda}}\leqslant b_2N^\lambda$$
where b_1,b_2 are constants, $N\geqslant 1$.

Proof See [5].

The linearized operator of the nonlinear operator $F_\varepsilon(u)$ is
$$F_\varepsilon'(u)v\equiv v_{tt}-v_{xx}+\varepsilon(a_{11}v_{tt}+a_{12}v_{tx}+a_{22}v_{xx}+a_1v_t+a_2v_x+a_0v)$$
where $a_0=g_u(t,x,u_{tt},\cdots,u)$, $a_1=g_{u_t}$, $a_2=g_{u_x}$, $a_{11}=g_{u_{tt}}$, $a_{12}=g_{u_{tx}}$, $a_{22}=g_{u_{xx}}$.

Let $u\in U_{2+\lambda}$, then $a\in F_{2+\lambda}$. Now, we consider the following periodic boundary value problem for linearized equation
$$F_\varepsilon'(u)v=h(t,x)\tag{3.1}$$
$$u(t,0)=0,\quad u_x(t,1)+u_t(t,1)=0\tag{3.2}$$
$$u(t+2\pi,x)=u(t,x)\tag{3.3}$$

Lemma 3.3 *Let $\alpha+\beta\leqslant 4+\lambda$, $\lambda=0,1,\cdots,k$, $\beta\leqslant 3$, then*
$$\|\partial_t^\alpha\partial_x^\beta(a_{11}v_{tt})-a_{11}\partial_t^\alpha\partial_x^\beta v_{tt}\|_{L^2(Q)}+\|\partial_t^\alpha\partial_x^\beta(a_{12}v_{tx})-a_{12}\partial_t^\alpha\partial_x^\beta v_{tx}\|_{L^2(Q)}$$
$$+\|\partial_t^\alpha\partial_x^\beta(a_{22}v_{xx})-a_{22}\partial_t^\alpha\partial_x^\beta v_{xx}\|_{L^2(Q)}+\|\partial_t^\alpha\partial_x^\beta(a_1v_t)\|_{L^2(Q)}$$
$$+\|\partial_t^\alpha\partial_x^\beta(a_2v_x)\|_{L^2(Q)}+\|\partial_t^\alpha\partial_x^\beta(a_0v)\|_{L^2(Q)}$$

$$\leqslant C\sum_{\nu=0}^{\lambda}\| a \|_{F_{2+\nu}}\| v \|_{U_{\lambda-\nu}} \tag{3.4}$$

where C is a constant, $\| a \|_{F_{2+\nu}} = \| a_{11} \|_{F_{2+\nu}} + \cdots + \| a_0 \|_{F_{2+\nu}}$.

Proof We shall discuss only that case in which $\alpha+\beta=4+\lambda, \beta\leqslant 3$. We take the third term at left side of (3.4) for example. It can be written as a summation of the following terms

$$A = \partial_t^{\alpha_1}\partial_x^{\beta_1}a_{22}\partial_t^{\alpha_2}\partial_x^{\beta_2}v_{xx}$$

where $\alpha_1+\beta_1+\alpha_2+\beta_2=4+\lambda, \beta_1+\beta_2=3$ and $\alpha_1+\beta_1>0$.

In the following estimates Lemma 3.1 is applied frequently.

If $\beta_2=3$, then $\alpha_1+\alpha_2=1+\lambda, \alpha_2\leqslant\lambda$ and

$$\| A \|_{L^2(Q)} \leqslant C \| \partial_t^{\alpha_1}a_{22} \|_{H^3(Q)} \| \partial_t^{\alpha_2}v_{xx} \|_{H^3(Q)}$$
$$\leqslant C \| a_{22} \|_{F_{\alpha_1}} \| v \|_{U_{\alpha_2}}$$

If $\beta_2=2$, then $\alpha_1+\alpha_2\leqslant 2+\lambda, \beta_1\leqslant 1$ and $\alpha_2\leqslant 1+\lambda$, we have

$$\| A \|_{L^2(Q)} \leqslant C \| \partial_t^{\alpha_1}\partial_x^{\beta_1}a_{22} \|_{H^2(Q)} \| \partial_t^{\alpha_2}v_{xx} \|_{H^2(Q)}$$
$$\leqslant C \| a \|_{F_{\alpha_1}} \| v \|_{U_{(\alpha_2-1)^+}}$$

where

$$(\alpha_2-1)^+ = \begin{cases} \alpha_2-1, & \text{if } \alpha_2-1>0 \\ 0, & \text{if } \alpha_2-1\leqslant 0 \end{cases}$$

To the cases in which $\beta_2=1$ or 0, the discussion is similar.

For the other terms, we can obtain the similar estimates in the same way.

Theorem 3.1 *Let $u\in U_{2+k}, h\in F_{2+k}$ satisfy*

$$\| u \|_{U_{2+\lambda}} \leqslant b^{-1}N^\lambda, \quad \| h \|_{F_{2+\lambda}} \leqslant bN^\lambda, \quad \lambda=0,1,\cdots,k \tag{3.5}$$

for $N\geqslant 1$. Then there exists $\varepsilon_0>0$ such that if $\varepsilon\leqslant\varepsilon_0$, then the equation

$$F_\varepsilon'(u)v = h \tag{3.6}$$

admits a solution $v\in U_k$ satisfying

i) $\| v \|_{U_0} \leqslant b \| h \|_{F_2}$

ii) $\| v \|_{U_\lambda} \leqslant b^2N^\lambda, \quad \lambda=0,1,\cdots,k$

Proof From Lemma 3.2 we have $a\in F_{2+\lambda}$. And from Theorem 2.4 it follows that equation (3.6) admits a solution $v\in U_k$.

Now we want to prove the estimates i) and ii) in the statement of the theorem. Operating on both sides of the equation (3.6) by $\partial_t^\alpha(\alpha\leqslant 4+\lambda)$, we have

$$(\partial_t^\alpha v)_{tt} - (\partial_t^\alpha v)_{xx} + \varepsilon\partial_t^\alpha(a_{11}v_{tt} + \cdots + a_0v) = \partial_t^\alpha h \tag{3.7}$$

Writing

$$G^\alpha(x) = \int_0^{2\pi}(|\partial_t^\alpha v_t|^2 + |\partial_t^\alpha v_x|^2)dt$$

from

$$\iint_Q \partial_t^\alpha v_x[\partial_t^\alpha v_{tt} - \partial_t^\alpha v_{xx}]dxdt$$
$$= -\varepsilon\iint_Q \partial_t^\alpha v_x\partial_t^\alpha(a_{11}v_{tt} + \cdots + a_0v)dxdt + \iint_Q \partial_t^\alpha v_x\partial_t^\alpha h dxdt, \tag{3.8}$$

we can obtain

$$G^{\alpha}(1) - G^{\alpha}(x) = 2\varepsilon \int_x^1 \int_0^{2\pi} \partial_t^{\alpha} v_x \partial_t^{\alpha} (a_{11} v_{tt} + \cdots + a_0 v) dx dt$$
$$- 2 \int_x^1 \int_0^{2\pi} \partial_t^{\alpha} v_x \partial_t^{\alpha} h dx dt \tag{3.9}$$

Noting

$$\left| \int_x^1 \int_0^{2\pi} \partial_t^{\alpha} v_x a_{11} \partial_t^{\alpha} v_{tt} dx dt \right|$$
$$= \left| -\frac{1}{2} \int_0^{2\pi} a_{11} |\partial_t^{\alpha} v_t|^2 (t,1) dt + \frac{1}{2} \int_0^{2\pi} a_{11} (\partial_t^{\alpha} v_t)^2 (t,x) dt \right.$$
$$\left. + \int_x^1 \int_0^{2\pi} \left\{ \frac{1}{2} \partial_x a_{11} |\partial_t^{\alpha} v_t|^2 - \partial_t a_{11} \partial_t^{\alpha} v_x \partial_t^{\alpha} v_t \right\} dx dt \right|$$
$$\leqslant C(G^{\alpha}(1) + G^{\alpha}(x)) + \| a \|_{F_0} \iint_Q \{ |\partial_t^{\alpha} v_t|^2 + |\partial_t^{\alpha} v_x|^2 \} dx dt \tag{3.10}$$

from Lemma 3. 3 it follows

$$\left| \int_x^1 \int_0^{2\pi} \partial_t^{\alpha} v_t \partial_t^{\alpha} (a_{11} v_{tt}) dx dt \right|$$
$$\leqslant \left| \int_x^1 \int_0^{2\pi} \partial_t^{\alpha} v_x a_{11} \partial_t^{\alpha} v_{tt} dx dt \right| + \left| \int_x^1 \int_0^{2\pi} \partial_t^{\alpha} v_x [\partial_t^{\alpha} (a_{11} v_{tt}) - a_{11} \partial_t^{\alpha} v_{xx}] dx dt \right|$$
$$\leqslant C(G^{\alpha}(1) + G^{\alpha}(x)) + \| a \|_{F_0} (\| \partial_t^{\alpha} v_t \|_{L^2(Q)}^2 + \| \partial_t^{\alpha} v_x \|_{L^2(Q)}^2)$$
$$+ C \sum_{\nu=0}^{\lambda} \| a \|_{F_{2+\nu}} \| v \|_{U_{\lambda-\nu}} \| \partial_t^{\alpha} v_x \|_{L^2(Q)} \tag{3.11}$$

and

$$\left| \int_x^1 \int_0^{2\pi} \partial_t^{\alpha} v_x a_{12} \partial_t^{\alpha} v_{xt} dx dt \right|$$
$$= \frac{1}{2} \left| \int_x^1 \int_0^{2\pi} \{ \partial_t (a_{12} (\partial_x^{\alpha} v_x)^2) - \partial_t a_{12} |\partial_t^{\alpha} v_x|^2 \} dx dt \right|$$
$$\leqslant \| a \|_{F_0} \| \partial_t^{\alpha} v_x \|_{L^2(Q)}^2 \tag{3.12}$$

Using (3. 12) and Lemma 3. 3, we can obtain in the way similar to (3. 11)

$$\left| \int_x^1 \int_0^{2\pi} \partial_t^{\alpha} v_x \partial_t^{\alpha} (a_{12} v_{tx}) dx dt \right|$$
$$\leqslant \| a \|_{F_0} \| \partial_t^{\alpha} v_x \|_{L^2(Q)} + C \sum_{\nu=0}^{\lambda} \| a \|_{F_{2+\nu}} \| v \|_{U_{\lambda-\nu}} \| \partial_t^{\alpha} v_x \|_{L^2(Q)} \tag{3.13}$$

Using the formula for integration by parts, it is not difficult to obtain

$$\left| \int_x^1 \int_0^{2\pi} \partial_x^{\alpha} v_x a_{22} \partial_t^{\alpha} v_{xx} dx dt \right| \leqslant C(G^{\alpha}(1) + G^{\alpha}(x)) + \| a \|_{F_0} \| \partial_t^{\alpha} v_x \|_{L^2(Q)}^2$$

So using Lemma 3. 3 we have

$$\left| \int_x^1 \int_0^{2\pi} \partial_t^{\alpha} v_x \partial_t^{\alpha} (a_{22} v_{xx}) dx dt \right|$$
$$\leqslant C(G^{\alpha}(1) + G^{\alpha}(x)) + \| a \|_{F_0} \| \partial_t^{\alpha} v_x \|_{L^2(Q)}^2$$
$$+ C \sum_{\nu=0}^{\lambda} \| a \|_{F_{2+\nu}} \| v \|_{U_{\lambda-\nu}} \| \partial_t^{\alpha} v_x \|_{L^2(Q)} \tag{3.14}$$

And applying still use Lemma 3. 3 it is easy to prove

$$\left|\int_0^1\int_0^{2\pi}\partial_t^\nu v_x\partial_t^\nu(a_1v_t+a_2v_x+a_0v)dxdt\right|$$

$$\leqslant C\sum_{\nu=0}^{\lambda}\|a\|_{F_{2+\nu}}\|v\|_{U_{\lambda-\nu}}\|\partial_t^\nu v_x\|_{L^2(Q)} \tag{3.15}$$

From (3. 9), (3. 11), (3. 13), (3. 14) and (3. 15) we have

$$\begin{aligned}G^a(1)-G^a(x)\geqslant &-C\varepsilon(G^a(1)+G^a(x))\\ &-\varepsilon\|a\|_{F_0}(\|\partial_t^\nu v_t\|^2_{L^2(Q)}+\|\partial_t^\nu v_x\|^2_{L^2(Q)})\\ &-C\varepsilon\sum_{\nu=0}^{\lambda}\|a\|_{F_{2+\nu}}\|v\|_{U_{\lambda-\nu}}\|\partial_t^\nu v_x\|_{L^2(Q)}\\ &-2\|\partial_t^\nu h\|_{L^2(Q)}\|\partial_t^\nu v_x\|_{L^2(Q)}\end{aligned} \tag{3.16}$$

Integrate (3. 16) from 0 to 1 it follows

$$\begin{aligned}&\|\partial_t^\nu v_t\|^2_{L^2(Q)}+\|\partial_t^\nu v_x\|^2_{L^2(Q)}\\ &\leqslant(1+C\varepsilon)G^a(1)+C\varepsilon(\|\partial_t^\nu v_t\|^2_{L^2(Q)}+\|\partial_t^\nu v_x\|^2_{L^2(Q)})\\ &\quad+\varepsilon\|a\|_{F_0}(\|\partial_t^\nu v_t\|^2_{L^2(Q)}+\|\partial_t^\nu v_x\|^2_{L^2(Q)})\\ &\quad+C\varepsilon\sum_{\nu=0}^{\lambda}\|a\|_{F_{2+\nu}}\|v\|_{U_{\lambda-\nu}}\|\partial_t^\nu v_x\|_{L^2(Q)}\\ &\quad+2\|\partial_t^\nu h\|_{L^2(Q)}\|\partial_t^\nu v_x\|_{L^2(Q)}\end{aligned} \tag{3.17}$$

Moreover, it is clear that

$$\iint_Q\partial_t^\nu v_t(\partial_t^\nu v_{tt}-\partial_t^\nu v_{xx})dxdt$$

$$=-\varepsilon\iint_Q\partial_t^\nu v_t\partial_t^\nu(a_{11}v_{tt}+a_{12}v_{tx}+\cdots+a_0v)dxdt+\iint_Q\partial_t^\nu v_t\partial_t^\nu h dxdt \tag{3.18}$$

Using the formula for integration by parts, we have

$$\iint_Q\partial_t^\nu v_t(\partial_t^\nu v_{tt}-\partial_t^\nu v_{xx})dxdt=\int_0^{2\pi}|\partial_t^\nu v_t(t,1)|^2dt \tag{3.19}$$

In the same way, we can obtain the following estimates

$$\begin{aligned}\left|\iint_Q\partial_t^\nu v_t\partial_t^\nu(a_{11}v_{tt})dxdt\right|\leqslant&\|a\|_{F_0}\|\partial_t^\nu v_t\|^2_{L^2(Q)}\\ &+C\sum_{\nu=0}^{\lambda}\|a\|_{F_{2+\nu}}\|v\|_{U_{\lambda-\nu}}\|\partial_t^\nu v_t\|_{L^2(Q)}\end{aligned} \tag{3.20}$$

$$\begin{aligned}\left|\iint_Q\partial_t^\nu v_t\partial_t^\nu(a_{12}v_{tx})dxdt\right|\leqslant&CG^a(1)+\|a\|_{F_0}\|\partial_t^\nu v_t\|^2_{L^2(Q)}\\ &+C\sum_{\nu=0}^{\lambda}\|a\|_{F_{2+\nu}}\|v\|_{U_{\lambda-\nu}}\|\partial_t^\nu v_t\|_{L^2(Q)}\end{aligned} \tag{3.21}$$

$$\left|\iint_Q\partial_t^\nu v_t\partial_t^\nu(a_{22}v_{xx})dxdt\right|\leqslant C(G^a(1)+\|\partial_t^\nu v_t\|^2_{L^2(Q)}+\|\partial_t^\nu v_x\|^2_{L^2(Q)})$$

$$+ C\sum_{\nu=0}^{\lambda} \| a \|_{F_{2+\nu}} \| v \|_{U_{\lambda-\nu}} \| \partial_t^\alpha v_t \|_{L^2(Q)} \tag{3.22}$$

and

$$\left| \iint_Q \partial_t^\alpha v_t \partial_t^\alpha (a_1 v_t + a_2 v_x + a_0 v) dx dt \right|$$

$$\leqslant C\sum_{\nu=0}^{\lambda} \| a \|_{F_{2+\nu}} \| v \|_{U_{\lambda-\nu}} \| \partial_t^\alpha v_t \|_{L^2(Q)} \tag{3.23}$$

From (3.18)—(3.23) we have

$$\begin{aligned}(1 - C\varepsilon)G^\alpha(1) \leqslant & C\varepsilon(\| \partial_t^\alpha v_t \|^2_{L^2(Q)} + \| \partial_t^\alpha v_x \|^2_{L^2(Q)}) \\ & + \varepsilon \| a \|_{F_0} \| \partial_t^\alpha v_t \|^2_{L^2(Q)} \\ & + C\varepsilon \sum_{\nu=0}^{\lambda} \| a \|_{F_{2+\nu}} \| v \|_{U_{\lambda-\nu}} \| \partial_t^\alpha v_t \|_{L^2(Q)} \\ & + \| \partial_t^\alpha h \|_{L^2(Q)} \| \partial_t^\alpha v_t \|_{L^2(Q)} \end{aligned} \tag{3.24}$$

If we take ε sufficiently small, from (3.17) and (3.24) it follows

$$\| \partial_t^\alpha v_t \|^2_{L^2(Q)} + \| \partial_t^\alpha v_x \|^2_{L^2(Q)}$$

$$\leqslant C\varepsilon \sum_{\nu=0}^{\lambda} \| a \|^2_{F_{2+\nu}} \| v \|^2_{U_{\lambda-\nu}} + \| \partial_t^\alpha h \|^2_{L^2(Q)} \tag{3.25}$$

Now we estimate the derivatives of v with respect to x. Differentiating equation (3.1) with respect to t and x, we have

$$\begin{aligned}\partial_t^\alpha \partial_x^\beta v = & \partial_t^\alpha \partial_x^{\beta-2} v_{tt} + \varepsilon \partial_t^\alpha \partial_x^{\beta-2} (a_{11} v_{tt} + a_{12} v_{tx} \\ & + a_{22} v_{xx} + a_1 v_t + a_2 v_x + a_0 v) - \partial_t^\alpha \partial_x^{\beta-2} h \\ & \alpha + \beta \leqslant 5 + \lambda, 2 \leqslant \beta \leqslant 5 \end{aligned} \tag{3.26}$$

Use Lemma 3.3, from (3.26) it is not difficult to estimate

$$\begin{aligned}\| \partial_t^\alpha \partial_x^\beta v \|_{L^2(Q)} \leqslant & \| \partial_t^\alpha \partial_x^{\beta-2} v_{tt} \|_{L^2(Q)} + \varepsilon \sum_{\nu=0}^{\lambda} \| a \|_{F_{2+\nu}} \| v \|_{U_{\lambda-\nu}} \\ & + \| \partial_t^\alpha \partial_x^{\beta-2} h \|_{L^2(Q)}, \quad \alpha + \beta \leqslant 5 + \lambda, 2 \leqslant \beta \leqslant 5 \end{aligned} \tag{3.27}$$

Note $v=0$ at $x=0$, $\| v \|_{L^2(Q)}$ can be dominated by $\| v_x \|_{L^2(Q)}$. So using (3.27) successively, we have

$$\begin{aligned}\| v \|_{U_\lambda} \leqslant & C(\| \partial_t^\alpha v_t \|_{L^2(Q)} + \| \partial_t^\alpha v_x \|_{L^2(Q)} \\ & + \varepsilon \sum_{\nu=0}^{\lambda} \| a \|_{F_{2+\nu}} \| v \|_{U_{\lambda-\nu}}) + \| h \|_{F_\lambda} \end{aligned} \tag{3.28}$$

From (3.25) and (3.28) it follows

$$\| v \|_{U_\lambda} \leqslant C\varepsilon \sum_{\nu=0}^{\lambda} \| a \|_{F_{2+\nu}} \| v \|_{U_{\lambda-\nu}} + C \| h \|_{F_{1+\lambda}} \tag{3.29}$$

From above inequality, the estimates i) and ii) are obtained immediately.

References

[1] Rabinowitz P. H., Periodic solutions of nonlinear hyperbolic partial differential equations Ⅱ, *Comm. Pure Appl. Math.*, **22**(1969), 15—39.

[2] Moser J., A new technique for the construction of solutions of nonlinear differential equa-

tions, *Proc. Nat. Acad. Sc. U. S. A.*, 47(1961), 1824—1831.

[3] Qin Tiehu, The global smooth solutions of second order quasilinear hyperbolic equations with dissipative boundary conditions, *Chin. Ann. of Math.*, 9B(3)(1988), 252—269.

[4] Mizohata S., The Theory of Partial Differential Equations, Cambridge University Press, 1973.

[5] Petzeltova H. and Stedry H., Time-periodic solutions of telegraph equations in n spatial variables, *Casopis pro pesto vani matematiky*, roc. 109(1984).