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RADIAL SOLUTIONS OF FREE BOUNDARY PROBLEMS FOR DEGENERATE PARABOLIC EQUATIONS

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Abstract In this paper we are devoted to the free boundary problem

$$\begin{cases} u_t = \Delta A(u) & (x,t) \in G_{\Gamma,T} \\ u(x,0) = \varphi(x) & x \in G_0 \\ u|_\Gamma = 0 \\ (\frac{\partial A(u)}{\partial x_i} \nu_i + \psi(x)\nu_t)|_\Gamma = 0 \end{cases}$$

where $A'(u) \geqslant 0$. Under suitable assumptions we obtain the existence and uniqueness of global radial solutions for $n=2$ and local radial solutions for $n \geqslant 3$.

Key Words High dimensions; degenerate parabolic; free boundary.
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1. Introduction

This paper is devoted to the following free boundary problem

$$\begin{cases} u_t = \Delta A(u) & (x,t) \in G_{\Gamma,T} \\ u(x,0) = \varphi(x) & x \in G_0 \\ u|_\Gamma = 0 \\ (\frac{\partial A(u)}{\partial x_i} \nu_i + \psi(x)\nu_t)|_\Gamma = 0 \end{cases} \tag{1.1}$$

where G_0 is a domain in R^n, $A'(u) \geqslant 0$, Γ is a surface in $R^n \times (0,T)$, $G_{\Gamma,T}$ is the domain bounded by G_0, Γ, and $\{t=T\}$, and $(\nu_x;\nu_t)$ is the normal to Γ.

The problem (1.1) comes from the analysis of the structure of discontinuous solutions for the equation $u_t = \Delta A(u)$ (see [7]). We also remark that the free boundary problem (1.1) is, in its form, the so-called Stefan problem studied by many authors. The difference between the problem (1.1) and Stefan problem is the degenerality in (1.1).

In the case $n=1$, we have dealt thoroughly with the problem (1.1). Under very general assumptions on $A(u)$, φ and ψ, we proved the existence and uniqueness of (1.1) and discussed the smoothness of free boundary. We obtained the necessary and sufficient condition for the free boundary in C^1 (see [4]).

We will restrict our attention to the problem (1.1) for $n \geqslant 2$ in this paper and only discuss a special case that can be reduced to a one-dimensional problem. The fundamental assumptions are

(H) $A(u)=u^m$ ($m>1$ is a constant), $G_0=B_1(O)$ is the unit ball in R^n, $\varphi(x)=\varphi(r)$, $\psi(x)=\psi(r)$, $r=(x_1^2+\cdots+x_n^2)^{1/2}$.

If the solution u has the form $u(x,t)=u(r,t)$ and Γ is determined by the function $r=\lambda(t)$ ($\lambda(0)=1$), then as a function of (r,t), (u,λ) satisfies

$$\begin{cases} u_t = u^m_{rr} + \dfrac{n-1}{r}u^m_r & 0<r<\lambda(t), 0<t<T \\ u(r,0)=\varphi(r), & 0<r<1 \\ u(\lambda(t),t)=0, & 0<t<T \\ u^m_r(\lambda(t),t)=\psi(\lambda(t))\lambda'(t), & 0<t<T \end{cases} \tag{1.2}$$

It is worth remarking that when $n=1$, the problem (1.2) has only one kind of degenerality, but for $n\geqslant 2$, besides the degenerality of u^m, there is an irregular factor $1/r$, which results in the important difference between $n=1$ and $n\geqslant 2$.

To solve (1.2), we introduce the transform:

$$y=\frac{r^{2-n}}{n-2}\quad(n>2),\qquad y=-\ln r\quad(n=2)$$

Set $v(y,t)=u(r,t)$, then $v(y,t)$ satisfies

$$\begin{cases} v_t = g_n(y)v^m_{yy}, & \lambda_n(t)<y<\infty, 0<t<T \\ v(y,0)=\varphi_n(y), & a_n<y<\infty \\ v(\lambda_n(t),t)=0, & 0<t<T \\ v^m_y(\lambda_n(t),t)=\psi_n(\lambda_n(t))\lambda_n'(t), & 0<t<T \end{cases} \tag{1.3}$$

where for $n>2$,

$$g_n(y)=(n-2)^{2(n-1)/(n-2)}y^{2(n-1)/(n-2)},\ \varphi_n(y)=\varphi(((n-2)y)^{1/(2-n)})$$

$$a_n=\frac{1}{n-2},\ \lambda_n(t)=\frac{\lambda(t)^{n-2}}{n-2}$$

$$\psi_n(y)=(n-2)^{-2(n-1)/(n-2)}\psi(((n-2)y)^{1/(2-n)})y^{-2(n-1)/(n-2)},\ 0<y<a_n$$

and for $n=2$,

$$g_n(y)=e^{2y},\quad \varphi_n(y)=\varphi(e^{-y}),\ a_n=0$$
$$\lambda_n(t)=-\ln\lambda(t),\quad \psi_n(y)=e^{-2y}\psi(e^{-y})$$

The paper is arranged as follows: In Section 2, we study the existence, uniqueness and regularity of the solutions of the problem (1.2). We prove that if φ and ψ satisfy suitable conditions, then for $n=2$, the problem (1.2) has a unique soluiton u for any $T>0$, and for $n\geqslant 3$ there exists a constant $t_n>0$ such that the problem (1.2) has a (unique) solution in $(0,t_n)$. In Section 3, we turn the results for the problem (1.2) to (1.1). The key to this procedure is to prove the following conclusion

$$\lim_{y\to\infty}(g_n(y))^{\alpha}\int_0^1|u^m_y(y,s)|ds=0$$

where $\alpha<n/(2(n-1))$ ($n\geqslant 2$), and u is the solution of the problem (1.2). The uniqueness of the problem (1.1) can be obtained as a consequence of a result due to Brézis and Crandall [1].

Nevertheless we do not obtain the condition for the free boundary in C^1 for $n\geqslant 2$.

2. Existence, Uniqueness and Regularity of the Problem (1.2)

Set $f(x)=((n-2)(x+a_n))^{2(n-1)/(n-2)}$ $(n>2)$ and $f(x)=e^{2x}$ $(n=2)$, $\lambda(t)=\lambda_n(t)-a_n(n\geqslant 2)$. Then (1.2) becomes

$$\begin{cases} u_t=f(x+\lambda(t))u_{xx}^m+\lambda'(t)u_x & 0<x<\infty, 0<t<T \\ u(x,0)=\varphi(x) & 0<x<\infty \\ u(0,t)=0 & 0<t<T \\ u_x^m(0,t)=\psi(\lambda(t))\lambda'(t) & 0<t<T \end{cases} \tag{2.1}$$

where $u(x,t)=v(x+\lambda(t),t)$. We assume that φ,ψ denote general functions in (2.1).

If (u,λ) is a classical solution of (2.1), then we get, from (2.1), that

$$\begin{aligned} \Psi(\lambda(t)) = & \int_0^x \frac{x-y}{x}(u(y,t)-\varphi(y))dy-\frac{1}{x}\int_0^t f(x+\lambda(s))u^m(x,s)ds \\ & +\frac{2}{x}\int_0^t\int_0^x f'(y+\lambda(s))u^m(y,s)dyds \\ & -\frac{1}{x}\int_0^t\int_0^x (x-y)f''(y+\lambda(s))u^m(y,s)dyds \\ & -\frac{1}{x}\int_0^t\int_0^x \lambda'(s)u(y,s)dyds \end{aligned} \tag{2.2}$$

where $\Psi(x)=-\int_0^x f(s)\psi(s)ds, \forall\ x>0$. Thus we have the following

Definition 2.1 *A pair of functions (v,λ) is said to be a solution of the problem (1.2) if (u,λ), defined in (2.1), is a solution of (2.1), that is, the following conditions are fulfilled:*

1) $u\geqslant 0, u\in C^0(\overline{Q}_{0,T}\cap[\varepsilon,T])\cap L^\infty(\overline{Q}_{0,T})$ *and* $u(0,t)=0$

2) $\lambda(0)=0,\quad \lambda'\leqslant 0, \lambda\in W^{1,1}[0,T]$

3) (u,λ) *satisfied*

$$\iint_{Q_{r,T}} (u^m(f\gamma)_{xx}+u\gamma_t-\lambda' u\gamma_x)dxdt = -\int_0^r \varphi(x)\gamma(x,0)dx+\int_0^T fu^m\gamma_x\Big|_0^r dt \tag{2.3}$$

where $\gamma\in C^{2,1}(\overline{Q}_{r,T}), \gamma(x,T)=\gamma(0,t)=\gamma(r,t)=0, Q_{r,T}=(0,r)\times(0,T)$ *and* $r>0$ *is arbitrary.*

4) (2.3) *is valid.*

Suppose that φ and ψ satisfy the conditions

$(\mathrm{H})_1$: $\quad 0\leqslant\varphi\leqslant K_1, \quad |(\varphi^m)'|\leqslant K_2$

$(\mathrm{H})_2$: $\quad \psi$ is measurable and $-M_0\leqslant\psi\leqslant-\varepsilon_0$

where K_1, K_2, M_0 and ε_0 are constants.

For given $\lambda\in W^{1,1}[0,T]$, We consider the first value problem

$$u_t=f(x+\lambda(t))u_{xx}^m+\lambda'(t)u_x \qquad 0<x<\infty,\quad 0<t<T \tag{2.4}$$

$$u(x,0)=\varphi(x) \qquad 0<x<\infty \tag{2.5}$$

$$u(0,t)=0 \qquad 0<t<T \tag{2.6}$$

A function u is said to be a solution of (2.4)—(2.6), if (u,λ) satisfies the conditions 1)—3) in Definition 2.1.

Now consider the regularized problem:

$$(\mathrm{P})_k \qquad \begin{cases} u_t = f(x+\lambda_k(t))(a_k(u)u_x)_x + \lambda_k'(t)u_x & 0<x<k, \quad 0<t<T \\ u(x,0) = \varphi_k(x) & 0<x<k \\ u(0,t) = \varepsilon_k & 0<t<T \\ u(k,t) = \varphi_k(k) & 0<t<T \end{cases}$$

where $a_k, \varphi_k, \varepsilon_k$ satisfy

$$\begin{cases} a_k(u) = \begin{cases} mu^{m-1} & u \geqslant \varepsilon_k \\ \text{smoothly connected} & \varepsilon_k/2 < u < \varepsilon_k \\ \varepsilon_k/2 & u < \varepsilon_k/2 \end{cases} \\ \varepsilon_k \downarrow 0, \varphi_k \in C^\infty[0,\infty), \varphi_k(0) = \varepsilon_k, \varepsilon_k \leqslant \varphi_k(x) \\ |(\varphi_k^m)'| \leqslant 2K_2, \varphi_k^{(p)}(0) = \varphi_k^{(p)}(k) = 0 (p = 1,2,\cdots) \\ \text{and } \varphi_k \to \varphi \text{ uniformly in every compact subset of } [0,\infty). \\ \lambda_k \in C^\infty[0,T], \lambda_k' \leqslant 0, \lambda_k(0) = 0 \text{ and } \|\lambda_k - \lambda\|_{W^{1,1}} \to 0 \end{cases} \tag{2.7}$$

From [2] or [6] and the well-known maximum principle, one easily gets that

Lemma 2.1 *The problem* $(\mathrm{P})_k$ *admits a unique solution* u_k *such that*

1) $\varepsilon_k \leqslant u_k \leqslant K_1 + \varepsilon_k$ *and thus* $u_{kt} = f(x+\lambda_k(t))u_{kxx}^m + \lambda_k'(t)u_{kx}$

2) $0 \leqslant u_{kx}^m(0,t) \leqslant 2K_2$ *and there exists a constant* C *independent of* k *such that*

$$|u_{kx}^m(x,t)| \leqslant C, \quad (x,t) \in [0,k] \times [0,T]$$

then there is a constant C' *depending only on* X *and* M_0 *such that*

$$|u_k^m(x,t) - u_k^m(y,s)| \leqslant C'(|x-y|^2 + |t-s|)^{1/2}$$

for $(x,t),(y,s) \in [0,x] \times [0,T]$.

Proof The assertion 1) is obtained immediately from [2,6]. The second one can be proved by using the similar method in the proof of Lemma 2.1 in [4].

Proposition 2.1 *The problem (2.4), (2.5) and (2.6) admits a unique solution* $u \in C_{\mathrm{loc}}^\alpha(\overline{Q}_{0,T})$ *and* u^m *is uniformly Lipschitz continuous in* x.

Proof The existence of solution can be obtained from 1) and 2) of Lemma 2.1. $u \in C_{\mathrm{loc}}(\overline{Q}_{0,T})$ can be proved from 2) and the method in [3] or [5]. Below we only prove the uniqueness.

Suppose v is another solution of (2.4)—(2.6). Then we have

$$\iint\limits_{Q_{\tau,T}} ((u^m - v^m)(f\gamma)_{xx} + (u-v)\gamma_t - \lambda'(u-v)\gamma_x)dxdt$$

$$= \int_0^T f(u^m - v^m)\gamma_x \Big|_0^r dt \tag{2.8}$$

where $Q_{r,T}$ and γ are given in 3) in Definition 2.1. Set

$$a(x,t) = m\int_0^1 (\theta u + (1-\theta)v)^{m-1} d\theta$$

Then (2.8) can be rewritten as

$$\iint\limits_{Q_{r,T}} (u-v)(a(x,t)(f\gamma)_{xx} + \gamma_t - \lambda'\gamma_x)dxdt$$

$$= \int_0^T f(u^m - v^m)\gamma_x \Big|_0^r dt \tag{2.9}$$

For any $h \in C_0^\infty(Q_{r,T})$, consider the problem

$$\begin{cases} a_k(x,t)(f\gamma)_{xx} + \gamma_t - \lambda_k'(t)\gamma_x = h \\ \gamma|_{t=T} = \gamma|_{x=0,r} = 0 \end{cases} \tag{2.10}$$

where $a_k \in C^\infty$, $a_k \geqslant a$ and $a_k \geqslant 1/k$, $|a_k - a| \to 0$ uniformly on every compact subset in $[0,\infty)\times[0,T]$, $\lambda_k \in C^\infty$ and $\|\lambda_k - \lambda\|_{W^{1,1}} \to 0$.

Set $w = f\gamma$. We deduce that w satisfies

$$\begin{cases} fa_k w_{xx} + w_t - \lambda_k'(t)w_x = fh \\ w|_{t=T} = w|_{x=0,r} = 0 \end{cases} \tag{2.11}$$

The problem (2.11) has a unique solution $w_{r,k} \in C^{2,1}(\overline{Q}_{r,T})$ (see [10]). Moreover $w_{r,k}$ has the following properties:

P_1: $|w_{r,k}| \leqslant \max|fh| \equiv M_1$ in $Q_{r,T}$ and $|w_{r,k}| \leqslant M_1(1+x)^{-\sigma}$ $(\sigma > 1)$

P_2: $|\frac{\partial}{\partial x} w_{r,k}| \leqslant M_2$, and $|\frac{\partial}{\partial x} w_{r,k}(r,t)| \leqslant M_2 r^{-\sigma}$

P_3: $\int_0^T \int_0^r \left(\frac{\partial}{\partial x} w_{r,k}\right)^2 + \int_0^T \int_0^r a_k (w_{r,k})_{xx}^2 \leqslant M_3$

where M_1, M_2, M_3 and σ are constants independent of k. In fact, the properties P_1 and P_2 can be proved as done in [8], and the property P_3 can be obtained by multiplying $(w_{r,k})_{xx}$ on the both sides of the equation in (2.11) and then integrating the resulting identity on $[0,r]\times[0,T]$.

Now set $\gamma_{r,k} = w_{r,k}/f$, then $\gamma_{r,k} \in C^{2,1}(\overline{Q}_{r,T})$ is a solution of (2.10). Substituting $\gamma_{r,k}$ into (2.9), we deduce that

$$\Big|\iint_{Q_{r,T}} (u-v)h\,dx\,dt\Big| \leqslant \|\sqrt{a_k}\, w_{r,k}\|_{L^2(Q_{r,T})} \cdot \|a_k - a\|_{L^\infty(Q_{r,T})}$$

$$+ C(M_1, M_2, M_3) r \int_0^T |\lambda_k' - \lambda'| + T\sup|u^m - v^m| M_2 r^{-\sigma}$$

$$\longrightarrow 0$$

Let $k \to \infty$ first and then $r \to \infty$. The proposition is proved.

The following theorem gives the existence and uniqueness of the solutions of (2.1).

Theorem 2.1 *Suppose that $(H)_1, (H)_2$ hold. Then, when $n = 2$, the problem (2.1) admits a solution in $(0,T)$ for any $T > 0$; when $n \geqslant 3$, there exists $t_n > 0$, such that the problem (2.1) admits a solution in $(0, t_n)$. For $n \geqslant 2$, the solution of (2.1) is unique.*

Proof The uniqueness of solutions can be proved by using the similar way in the proof of Proposition 2.1. Here we only prove the existence.

Define the compact and convex set in C^0 as follows:

$$V = \{\lambda; \lambda \in W^{1,\infty}[0,t_0], \lambda(0) = 0, \lambda(t) \text{ nonincreasing},$$
$$|\lambda'(t)| \leqslant 2K_2 f(0)/\varepsilon_0 \quad \text{and} \quad -\tilde{a}_n \leqslant \lambda \leqslant 0\}$$

where $\tilde{a}_n = 1/(n-2)$ $(n > 2)$ and $\tilde{a}_2 = \infty$, $t_0 > 0$ is determined later.

For any $\lambda \in V$, denote by u the solution of (2.4)–(2.6) corresponding to λ. Suppose that u_k is the solution of $(P)_k$ which converges to u. Let

$$F_k(t) = -\int_0^t f(\lambda_k(s)) u_{kx}^m(0,s) ds$$

Then $F_k' \leqslant 0$, $|F_k'| \leqslant K_2 f(0)$ (see Lemma 2. 1), and

$$\begin{aligned}
F_k(t) = & \frac{1}{x}\int_0^x (x-y)(u_k(y,t) - \varphi_k(y)) dy - \frac{1}{x}\int_0^t f(x+\lambda_k(\tau)) u_k^m(x,\tau) d\tau \\
& + \frac{\varepsilon_n}{x}\int_0^t f(\lambda_k(\tau)) d\tau + \frac{1}{x}\int_0^t\int_0^x f'(y+\lambda_k(\tau)) u_k^m(y,\tau) dy d\tau \\
& + \frac{1}{x}\int_0^t\int_0^x f'(y+\lambda_k(\tau)) u_k^m(y,\tau) dy d\tau - \varepsilon_n \int_0^t f'(\lambda_k(\tau)) d\tau + \varepsilon_k \lambda_k(t) \\
& - \frac{1}{x}\int_0^t\int_0^x (x-y) f''(y+\lambda_k(\tau)) u_k^m(y,\tau) dy d\tau \\
& - \frac{1}{x}\int_0^t\int_0^x \lambda_k'(\tau) u_k(y,\tau) dy d\tau \qquad (2.12)
\end{aligned}$$

Here and below, n denotes the dimension and k the index of sequence.

Define the operator $T^*: V \to C^0[0,t_0]$ as follows

$$\begin{aligned}
\Psi(T^*\lambda)(t) = & \frac{1}{x}\int_0^x (x-y)(u(y,t) - \varphi(y)) dy - \frac{1}{x}\int_0^t f(x+\lambda(\tau)) u^m(x,\tau) d\tau \\
& + \frac{2}{x}\int_0^t\int_0^x f'(y+\lambda(\tau)) u^m(y,\tau) dy d\tau \\
& - \frac{1}{x}\int_0^t\int_0^x (x-y) f''(y+\lambda(\tau)) u^m(y,\tau) dy d\tau \\
& - \frac{1}{x}\int_0^t\int_0^x \lambda'(\tau) u(y,\tau) dy d\tau \qquad (2.13)
\end{aligned}$$

As it was done in the proof of Theorem 2. 2 in [4], from $\lim\limits_{k\to\infty} F_k(t) = \Psi(T^*\lambda)$ (see (2. 12)) and the independence to x of F_k we know that T^* is well defined.

If $n=2$, for any $t_0 = T$, it is evident that $T^*V \subset V$. If $n \geqslant 3$, from (2. 13) and Proposition 2. 1 one has

$$\begin{aligned}
|\Psi(T^*\lambda)(t)| & \leqslant 2K_1 x + \frac{f(x)K_1^m}{x} t + 2K_1^m f'(x) t + f''(x) x K_1^m t + K_1 \lambda(t) \\
& \leqslant 2K_1 x + \left[\frac{f(x)K_1^m}{x} + 2K_1^m f'(x) + x K_1^m f''(x) + \frac{K_1^2 f(0)}{\varepsilon_0}\right] t
\end{aligned}$$

where $x > 0$ is arbitrary. Therefore there are constants $C_1 - C_5$, depending only on K_1 and n, such that

$$|\Psi(T^*\lambda)(t)| \leqslant (C_1 + C_2 t) x + C_3 x^{-\nu-1} t + C_4 t/x + C_5 t$$

where $\nu = 2(n-1)/(n-2)$. Choose x such that the right side of the above inequality becomes minimum. Then there exists a continuous function $f_0(t)$, depending only on K_1 and n, such that $f_0(0) = 0$, f_0^{-1} exists and is continuous, and

$$|\Psi(T^*\lambda)(t)| \leqslant f_0(t) + C_5 t$$

Thus for $\tilde{a}_n > 0$, choose $t_0 = t_n$ small enough such that

$$|\Psi(T^*\lambda)(t)| \leqslant \tilde{a}_n \qquad t \in (0, t_n), \lambda \in V$$

This shows that for $n \geqslant 3$, there is a $t_n > 0$ such that $T^*V \subset V$.

Now we can use the same method as in the proof of Theorem 2. 2 in [4] to prove that when $n \geqslant 2$, the operator T^* is continuous. And therefore T^* has a fixed point λ. This λ and the solution u of (2. 4)—(2. 6) corresponding to this λ give a solution of the problem (2. 1).

Similar to Corollary 2. 1 in [4], we have

Proposition 2. 2 *Suppose that* (φ_1, ψ_1) *and* (φ_2, ψ_2) *satisfy* $(H)_1$ *and* $(H)_2$. *Let* (u_1, λ_1) *and* (u_2, λ_2) *be the solutions of the problem* (2. 1) *corresponding to* (φ_1, ψ_1) *and* (φ_2, ψ_2), *respectively. If* $\varphi_1 \geqslant \varphi_2$ *and* $\psi_1 \geqslant \psi_2$, *then* $\lambda_1 \leqslant \lambda_2$ *and* $v_1 \geqslant v_2$, *where* $v_i(x,t) = u_i(x - \lambda_i(t), t)$, $i = 1, 2$.

In the following two propositions we give some properties of the solutions of (2. 1).

Proposition 2. 3 *Suppose that* (u, λ) *is the solution of* (2. 1). *Then for any rectangle* $A = (a,b) \times (c,d) \subset Q_{0,T}$, *there exists a constant* $C = C(b-a, d-c, c, K_1)$ *such that*

$$|u_x^{m-1}| \leqslant C \qquad \text{on } A$$

Thus $u \in C_{\mathrm{loc}}^{\alpha,\alpha/2}(Q_{0,T})$ *with* $\alpha = \min(1, 1/(m-1))$.

Proposition 2. 4 *Suppose that* (u, λ) *is the solution of* (2. 1). *Then* u_x^m *exists everywhere and* $u_x^m \in C^0(Q_{0,T})$. *Moreover* $u_x^m(x_0, t_0) = 0$ *if* $u(x_0, t_0) = 0$.

The proofs of the two propositions above are similar to those in Proposition 3. 1 and Proposition 3. 2 in [4]. Here we omit them.

Proposition 2. 5 *Suppose that* (u, λ) *is the solution of* (2. 1). *Then there exists a constant* C, *depending only on* K_1, *such that*

$$f(x + \lambda(t), t) u_{xx}^m \geqslant -C/t, \quad t > 0 \tag{2. 14}$$

in the sense of distribution. Thus the limit $\lim\limits_{x \to 0} u_x^m(x,t) \equiv u_x^m(0,t)$ *exists everywhere and*

$$u_x^m(0,t) = \psi(\lambda(t)) \lambda'(t) \qquad \text{a. e. in} \quad [0, T] \tag{2. 15}$$

Proof We use the notation in the proof of Theorem 2. 1. Let u_k be the solution of $(P)_k$ which converges to u. Then

$$u_{kt} = f(x + \lambda_k(t)) u_{kxx}^m + \lambda_k'(t) u_{kx}, \qquad 0 < x < k$$

Put $P_k = f(x + \lambda_k(t)) u_{kxx}^m$. Then P_k satisfies

$$P_{kt} = f(x + \lambda_k(t)) m w^a P_{kxx} + (2 m a w^{a-1} w_x f(x + \lambda_k(t)) + \lambda_k'(t)) P_{kx}$$
$$+ m a (a-1) w^{a-2} w_x^2 P_k + m a w^{a-1} P_k^2$$

where $w = u_k^m$, $a = (m-1)/m$. From this and Proposition 3. 3 in [4] it follows that

$$P_k(x,t) \geqslant -C/t, \qquad t > 0$$

for some $C = C(K_1)$. Therefore (2. 14) holds.

The relation (2. 15) is only a consequence of (2. 14) and Proposition 2. 4. In fact, (2. 14) shows that

$$u_x^m + \frac{C}{t} \int_0^x \frac{dy}{f(y + \lambda(t))} \tag{2. 16}$$

is increasing in x in the usual sense. Hence $\lim\limits_{x \to 0} u_x^m(x,t)$ exists everywhere. And (2. 15) now follows from (2. 2).

Proposition 2. 6 *Suppose that* (u, λ) *is the solution of* (2. 1). *Then for any* $s \in (0, T)$, *when* $a < n/2(n-1)$ $(n \geqslant 2)$, *we have*

$$\lim_{x \to \infty} (f(x))^a \int_s^t |u_x^m(x, \tau)| \, d\tau = 0 \tag{2. 17}$$

for $0<s<t<T$. *Further, we also have that for each* $t\in(0,T)$

$$\lim_{x\to\infty} u_x^m(x,t) = 0 \tag{2.18}$$

Proof As we have done in the proof of Proposition 2.5, we assume that u_k is the solution of $(P)_k$, $u_k\to u$ in $C^0_{loc}(Q_{0,T})$. Since

$$u_{kt} = f(x+\lambda_k(t))u^m_{kxx} + \lambda_k'(t)u_{kx} \tag{2.19}$$

we get that for $x,y\in(0,k)$

$$\int_0^t (u^m_{kx}(x,\tau) - u^m_{kx}(y,\tau))d\tau = -\int_0^t \lambda_k'(\tau)\left[\frac{u_k(x,\tau)}{f(x+\lambda_k(\tau))} - \frac{u_k(y,\tau)}{f(y+\lambda_k(\tau))}\right]d\tau + \int_y^x\left[\frac{u_k(z,t)}{f(z+\lambda_k(t))} - \frac{\varphi_k(z)}{f(z)}\right]dz$$

Letting $k\to\infty$, we obtain that for $x,y\in(0,\infty)$

$$\int_0^t (u^m_x(x,\tau) - u^m_x(y,\tau))d\tau = -\int_0^t \lambda'(\tau)\left(\frac{u(x,\tau)}{f(x+\lambda(\tau))} - \frac{u(y,\tau)}{f(y+\lambda(\tau))}\right)d\tau + \int_y^x\left(\frac{u(z,t)}{f(z+\lambda(t))} - \frac{\varphi(z)}{f(z)}\right)dz \tag{2.20}$$

From this and the definition of f it follows that

$$\varlimsup_{\substack{x\to\infty\\ y\to\infty}} \left|\int_0^t u^m_x(x,\tau)d\tau - \int_0^t u^m_x(y,\tau)d\tau\right|$$

$$\leqslant |\lambda(t)|K_1\left(\lim_{x\to\infty}\frac{1}{f(x+\lambda(t))} + \lim_{y\to\infty}\frac{1}{f(y+\lambda(t))}\right) + 2K_1\lim_{\substack{x\to\infty\\ y\to\infty}}\left|\int_y^x \frac{1}{f(z+\lambda(t))}dz\right| = 0$$

This shows that $\{u^m_x(x,s)ds\}$ satisfies Cauchy's rule in x; the function is also bounded from Lemma 2.1. Thus

$$\lim_{x\to\infty}\int_0^t u^m_x(x,s)ds = l(t)$$

exists everywhere in $(0,T)$.

If there is $t_0\in(0,T)$ such that $l(t_0)>0$, then there exists $X>0$, such that for $x\geqslant X$,

$$\int_0^{t_0} u^m_x(x,s)ds > \frac{1}{2}l(t_0)$$

Thus

$$\frac{\partial}{\partial x}\int_0^{t_0} u^m(x,s)ds > \frac{1}{2}l(t_0), \qquad x\geqslant X$$

and

$$K_1^m t_0 \geqslant \int_0^{t_0} u^m(x,s)ds > \frac{1}{2}l(t_0)(x-X) + \int_0^{t_0} u^m(X,s)ds$$

which is a contradiction. In the same way we can prove that there does not exist $t\in(0,T)$ such that $l(t)<0$. Thus $l(t)=0$ for all $t\in(0,T)$, i.e.

$$\lim_{x\to\infty}\int_0^t u^m_x(x,s)ds = 0, \qquad t\in(0,T) \tag{2.21}$$

On the other hand, from the definition of f it follows that

$$\int_0^{\infty}\frac{dy}{f(y+\lambda(t))}<\infty,\qquad t\in(0,T)$$

Therefore we see, from (2.16) and 2) in Lemma 2.1, that the function

$$u_x^m(x,t)+\frac{C}{t}\int_0^x\frac{dy}{f(y+\lambda(t))}$$

is non-decreasing in x in $(0,\infty)$ for each t. Hence $\lim\limits_{x\to\infty}u_x^m(x,t)$ exists for every $t\in(0,T)$, and the limit value is just zero by (2.21). Thus (2.18) is valid.

Below we prove (2.17). From (2.19) we have

$$|u_{kx}^m(x,t)-u_{kx}^m(y,t)|\leqslant\int_y^x\frac{|u_{kt}-\lambda_k'(t)u_{kz}(z,t)|}{f(z+\lambda_k(t))}dz\qquad(x\geqslant y)$$

Since $f(z+\lambda_k(t))u_{kzz}^m(z,t)\geqslant -C/t$, see the proof of Proposition 2.5, where $C=C(K_1)>0$, (2.19) shows that when $x\geqslant y$ and $t\geqslant s$,

$$\begin{aligned}
&\int_s^t|u_{kx}^m(x,\tau)-u_{kx}^m(y,\tau)|d\tau\\
&\leqslant\int_s^t\int_y^x\frac{u_{kt}(z,\tau)-\lambda_k'(\tau)u_{kz}(z,\tau)}{f(z+\lambda_k(\tau))}dzd\tau+\frac{2c}{s}\int_s^t\int_y^x\frac{dzd\tau}{f(z+\lambda_k(\tau))}\\
&=\int_y^x\left[\frac{u_k(z,t)}{f(z+\lambda_k(t))}-\frac{\varphi_k(z)}{f(z)}\right]dz-\int_0^t\lambda_k'(\tau)\frac{u_k(x,\tau)}{f(x+\lambda_k(\tau))}d\tau\\
&\quad+\int_0^t\lambda_k'(\tau)\frac{u_k(y,\tau)}{f(y+\lambda_k(\tau))}d\tau+\frac{2c}{s}\int_y^x\int_s^t\frac{dzd\tau}{f(z+\lambda_k(\tau))}\\
&\leqslant\int_y^x\left[\frac{u_k(z,t)}{f(z+\lambda_k(t))}-\frac{\varphi_k(z)}{f(z)}\right]dz-\int_0^t\lambda_k'(\tau)\frac{u_k(x,\tau)}{f(x+\lambda_k(\tau))}d\tau\\
&\quad+\frac{2c}{s}\int_y^x\int_s^t\frac{dzd\tau}{f(z+\lambda_k(\tau))}
\end{aligned}$$

since $\lambda_k'\leqslant 0$.

Letting $k\to\infty$, we get

$$\begin{aligned}
\int_s^t|u_x^m(x,\tau)-u_x^m(y,\tau)|d\tau&\leqslant\int_y^x\left(\frac{u(z,t)}{f(z+\lambda(t))}-\frac{\varphi(z)}{f(z)}\right)dz\\
&\quad-\int_0^t\lambda'(\tau)\frac{u(x,\tau)}{f(x+\lambda(\tau))}d\tau+\frac{2c}{s}\int_y^x\int_s^t\frac{dzd\tau}{f(z+\lambda(\tau))}
\end{aligned}$$

Then letting $x\to\infty$, and noting $\varphi,f\geqslant 0$, from (2.18) (in fact, the inequality said above implies the fact $\lim\limits_{x\to\infty}\int_s^t|u_x^m(x,s)|ds=0$) one gets that

$$\begin{aligned}
\int_s^t|u_x^m(y,\tau)|d\tau&\leqslant\int_y^{\infty}\frac{u(z,t)}{f(z+\lambda(t))}dz+\frac{2c}{s}\int_y^{\infty}\int_s^t\frac{dzd\tau}{f(z+\lambda(\tau))}\\
&\leqslant\left(K_1+\frac{2c}{s}t\right)\int_y^{\infty}\frac{dz}{f(z+\lambda(t))}
\end{aligned}$$

When $n=2$, $f(x)=e^{2x}$, and by L'Hospital rule, it follows that for any $\alpha<1$,

$$\begin{aligned}
0\leqslant\overline{\lim_{y\to\infty}}(f(y))^{\alpha}\int_s^t|u_x^m(y,\tau)|d\tau&\leqslant\left(K_1+\frac{2ct}{s}\right)\lim_{y\to\infty}e^{2\alpha y}\int_y^{\infty}\frac{dz}{e^{2z}e^{2\lambda(t)}}\\
&=e^{-2\lambda(t)}\left(K_1+\frac{2c}{s}t\right)\lim_{y\to\infty}\frac{1}{2\alpha}e^{-2(1-\alpha)y}=0
\end{aligned}$$

When $n\geqslant 3, f(x)=(x+a_n)^{2(n-1)/(n-2)}$, it follows that for any $a<n/(2(n-1))$,

$$0\leqslant\overline{\lim_{y\to\infty}}(f(y))^a\int_s^t|u_x^m(y,\tau)|d\tau$$

$$\leqslant\left(K_1+\frac{2c}{s}t\right)\lim_{y\to\infty}(y+a_n)^{2a(n-1)/(n-2)}\int_y^\infty\frac{dz}{(z+\lambda(t)+a_n)^{2(n-1)/(n-2)}}$$

$$=\left(K_1+\frac{2c}{s}t\right)\lim_{y\to\infty}\frac{n-2}{2a(n-1)}\frac{(y+a_n)^{2a(n-1)/(n-2)+1}}{(y+\lambda(t)+a_n)^{2(n-1)/(n-2)}}=0$$

Therefore when $a<n/(2(n-1)), n\geqslant 2$ we have

$$\lim_{y\to\infty}(f(y))^a\int_s^t|u_x^m(y,\tau)|d\tau=0$$

which shows that (2. 17) holds.

In the preceding discussions we assume that φ,ψ satisfy $(\mathrm{H})_1$ and $(\mathrm{H})_2$. The following theorems show that these conditions are not necessary and can be weakened.

Theorem 2. 2 *If ψ satisfies $(H)_2$ and φ satisfies*

$(H)_1'$: *$\varphi\geqslant 0$ is measurable and $\varphi\leqslant K_1$, then*

1) *for $n=2$, the problem (2. 1) admits a unique solution (u,λ) in $(0,T)$ for any $T>0$;*

2) *for $n\geqslant 3$, there exists $t_n>0$ such that the problem (2. 1) has a solution in $(0,t_n)$ and the solution is unique.*

In addition, Propositions 2. 2—2. 7 are still true except (2. 18). In Theorem 2. 2, (u,λ) is a solution if and only if (u,λ) satisfies Definition 2. 1.

Proof of Theorem 2. 2 The uniqueness may refer to Theorem 2. 3 in [4]. The last part of the theorem can be derived out from the following proof of the existence and Section 2 in [4].

Now we prove the existence. Choose a sequence φ_k which satisfies $(\mathrm{H})_1$ and is decreasing and

$$\|\varphi_k-\varphi\|_{L^1_{loc}[0,\infty)}\to 0$$

Suppose that (u_k,λ_k) is the solution of (2. 1) corresponding to (φ_k,ψ) whose existence-interval is $(0,t_{n,k})$, where $n=2, t_{n,k}=T$, when $n\geqslant 3$, from the proof of Theorem 2. 1 one has

$$|\Psi(\lambda_k(t))|\leqslant(C_1+C_2t)x+C_3x^{\gamma-1}t+C_4t/x+C_5t$$

where $x>0$ is arbitrary, C_1-C_5 are constants depending only on n, K_1 and $\gamma=2(n-1)/(n-2)$. Since $x>0$ is arbitrary, choosing $x=(C_4t/(C_1+C_2t))^{1/2}$ we have

$$|\lambda_k(t)|\leqslant\frac{2}{\varepsilon_0}((C_1+C_2t)C_4t)^{1/2}+\frac{C_3}{\varepsilon_0}(C_4/C_1)^{(\gamma-1)/2}t^{(\gamma-1)/2}+\frac{C_5}{\varepsilon_0}t\equiv f_0(t)$$

Clearly $f_0(t)$ is continuous and $f_0(0)=0$. Therefore there is a constant $t_0>0$ such that $f_0(t)<a_n, t\in[0,t_0]$. Thus $t_{n,k}\geqslant t_0>0$.

From Propositions 2. 2 and 2. 5 we see that for $\bar{u}_k(x,t)=u_k(x-\lambda_k(t),t)$ there hold

$$K_1\geqslant\bar{u}_1\geqslant\bar{u}_2\geqslant\cdots\geqslant\bar{u}_k\geqslant\cdots\geqslant 0$$

$$\lambda_1\leqslant\lambda_2\leqslant\lambda_3\leqslant\cdots\leqslant\lambda_k\leqslant\cdots\leqslant 0$$

and

$$u^m_{kx}(0,t) = \psi(\lambda_k(t))\lambda_k'(t) \quad \text{a. e. in} \quad [0,t_{n,k}] \tag{2.22}$$

Without loss of generality, we may assume that $\lim_{k\to\infty} t_{n,k} = t_*$, then $t_* \geqslant t_0 > 0$. Thus

$$\lambda(t) \equiv \lim_{k\to\infty} \lambda_k(t) \qquad \text{in} \quad [0,t_*] \tag{2.23}$$

$$u(x,t) = \lim_{k\to\infty} u_k(x,t), \qquad (x,t) \in Q_{0,t_*} \tag{2.24}$$

exist everywhere and the convergences are uniform in any compact subset in $\overline{Q}_{0,t_*} \cap [\varepsilon, t_*]$, by Propositions 2.3, 2.4 and 2.5. Moreover from (2.22) we have

$$0 \leqslant \psi(\lambda_k(t))\lambda_k'(t) = u^m_{kx}(0,t) \leqslant u^m_{kx}(1,t) + \frac{C(K_1)}{\varepsilon}(1 - \lambda_k(t))$$

$$\leqslant C_1 + \frac{C(K_1)}{\varepsilon}(1 + |\lambda_1(T)|) \equiv C, \quad t \in [\varepsilon, t_*]$$

Thus $|\lambda_k'(t)| \leqslant C/\varepsilon_0$ for $t \in [\varepsilon, t_*]$ a. e.. From this and $\lambda_k' \leqslant 0$, (2.23) one can prove λ' exists and $\lambda' \in L^1[0,t_*]$, $\lambda_k' \to \lambda'$ in $L^1[0,t_*]$. This shows that the convergence in (2.23) is uniform.

After the discussions above, we easily prove that (u,λ) is a solution, as done in the proofs in Remark 3.2 and Theorem 2.1 in [4].

Remark 2.1 If φ satisfies $(H)_1'$ and ψ is locally integrable in $(-\infty, 0]$ and $\psi(x) \leqslant -\varepsilon_0 < 0$, then

(1) when $n=2$ for any $T>0$, the problem (2.1) admits a unique solution in $(0, T)$;

(2) when $n \geqslant 3$, there exists $t_* > 0$ such that the problem (2.1) has a solution in $(0, t_*)$ and the solution is unique. In addition, Propositions 2.2—2.7 are still valid.

Theorem 2.3 *Suppose that φ satisfies $(H)_1'$ and ψ satisfies $(H)_2'$: ψ is locally integrable in $(-\infty, 0]$ and there exist $C>0$ and $a>-1$ such that*

$$\psi(x) \leqslant -C|x|^a, \qquad x < 0$$

Then the problem (2.1) admits a unique solution in $(0,T)$ for any $T>0$ when $n=2$; and when $n \geqslant 3$ there exists $t_*>0$ such that the problem (2.1) has a solution in $(0,t_*)$ and the solution is unique. In addition, Propositions 2.2—2.7 are still valid.

Proof (1) The proof for $n=2$.

By Remark 2.1, if we assume that $\psi_k(x) = \psi(x) - \varepsilon_k$ where $\{\varepsilon_k\}$ is a decreasing sequence tending to zero, then for any $T>0$ the problem (2.1) has a unique solution (u_k, λ_k), corresponding to (φ, ψ_k), which satisfies

$$0 \leqslant u_1(x - \lambda_1(t), t) \leqslant u_2(x - \lambda_2(t), t)$$

$$\leqslant \cdots \leqslant u_k(x - \lambda_k(t), t) \leqslant \cdots \leqslant K_1 \tag{2.25}$$

$$\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_k \geqslant \cdots \tag{2.26}$$

$$u^m_{kx}(0,t) = \psi_k(\lambda_k(t))\lambda_k'(t) \quad \text{a. e. in} \quad [0,T] \tag{2.27}$$

From (2.27), Propositions 2.4, 2.5 we see that there is a constant $C = C(K_1, \varepsilon) > 0$ such that

$$\psi(\lambda_k(t))\lambda_k'(t) \leqslant \psi_k(\lambda_k(t))\lambda_k'(t) = u^m_{kx}(0,t)$$

$$\leqslant u^m_{kx}(1,t) + \frac{C}{\varepsilon} \leqslant C \quad \text{a. e. in} \quad [\varepsilon, T] \tag{2.28}$$

Therefore

$$|\lambda_k(t)| \leqslant C(\varepsilon, K_1, \alpha) \qquad t \in [\varepsilon, T] \tag{2.29}$$

which shows (by (2.26)) that $\lambda(t) \equiv \lim_{k\to\infty} \lambda_k(t)$ exists everywhere in $[0,T]$. Hence (2.25) gives that limits

$$\lim_{k\to\infty} u_k(x,t) = u(x,t) \qquad \text{in} \quad Q_{0,T} \tag{2.30}$$

$$\lim_{k\to\infty} \lambda_k(t) = \lambda(t) \qquad \text{in} \quad [0,T] \tag{2.31}$$

all exist.

Again, from the assumption $(\mathrm{H})_2'$, for $t_1, t_2 \in [\varepsilon, T]$ there holds

$$C_1 \left| \int_{t_1}^{t_2} |\lambda_k^\alpha| \lambda_k'(\tau) d\tau \right| \leqslant \left| \int_{t_1}^{t_2} \psi(\lambda_k(\tau)) \lambda_k'(\tau) d\tau \right| \leqslant C|t_1 - t_2| \quad (C = C(K_1, \varepsilon))$$

Thus

$$\big| |\lambda_k(t_1)|^{\alpha+1} - |\lambda_k(t_2)|^{\alpha+1} \big| \leqslant C' |t_1 - t_2|$$

$$|\lambda_k(t_1) - \lambda_k(t_2)| \leqslant C |t_1 - t_2|^\beta, \quad t_1, t_2 \in [\varepsilon, T] \tag{2.32}$$

where $\beta = \min\{1, 1/(1+\alpha)\}$. This shows that the convergence in (2.31) is uniform in $[\varepsilon, T]$ for any $\varepsilon > 0$. In the same way we may prove that u is continuous in $\bar{Q}_{0,T} \cap [\varepsilon, T]$. Thus the convergence in (2.30) is also uniform in any compact subset in $\bar{Q}_{0,T} \cap [\varepsilon, T]$ for any $\varepsilon > 0$.

On the other hand, since λ_k and λ are non-increasing functions and $\lambda_k(0) = \lambda(0) = 0$, there exist t_k^* and $t_0 \in [0,T]$ such that $\lambda_k|_{[0,t_k^*]} = 0$, $\lambda_k < 0$ for $t \in (t_k^*, T]$ and $\lambda|_{[0,t_0]} = 0$, $\lambda < 0$ for $t \in (t_0, T]$. Note that t_k^* is decreasing as λ_k is. We conclude that $\lim_{k\to\infty} t_k^* = t_0$.

Choose $t^* \in (t_0, T]$ (if $t_0 = T$, Then $\lambda_k = \lambda = 0$ which is trivial). Then $\lambda(t^*) < 0$. Since $\lim_{k\to\infty} \lambda_k(t^*) = \lambda(t^*)$, there exists K such that $\lambda_k(t^*) < \frac{1}{2}\lambda(t^*)$ for $k > K$. And the monotonousness of λ_k shows that

$$\lambda_k(t) \leqslant \lambda_k(t^*) \leqslant \frac{1}{2}\lambda(t^*) < 0, \qquad t \geqslant t^*$$

Therefore, from $(\mathrm{H})_2'$ and (2.28) we obtain that

$$C\left(\frac{1}{2}|\lambda(t^*)|\right)^\alpha |\lambda_k'(t^*)| \leqslant C \qquad \text{a. e. for} \quad t \geqslant t^*$$

i. e.

$$|\lambda_k'(t)| \leqslant C_* \quad \text{a. e. in} \quad [t^*, T] \tag{2.32}$$

We conclude that there is a function $f_* \in L^1[t^*, T]$, $f_* \leqslant 0$ such that

$$\lambda_k' \xrightarrow{w} f_* \quad \text{in} \quad L^1[t^*, T]$$

Thus

$$\lambda_k(t) - \lambda_k(t^*) = \int_{t^*}^t \lambda_k'(s) ds$$

implies

$$\lambda(t) - \lambda(t^*) = \int_{t^*}^t f_*(s) ds$$

for $t \in [t^*, T]$. This shows that λ'exists a. e. in $[t^*, T]$ and $\lambda' = f_*$. It is easily seen that (2. 32)' is valid for λ'. Note the arbitrariness of t^* ($>t_0$) we see that λ' exists a. e. in $[t_0,T]$.

Secondly, by the formula

$$\lambda(t)-\lambda(t^*)=\int_{t^*}^{t}\lambda'(s)ds=-\int_{t^*}^{t}|\lambda'(s)|ds$$

and the continuity of λ (see (2. 32)), it follows that

$$-\int_{t_0}^{t}|\lambda'(s)|ds=\lambda(t)-\lambda(t_0)=\lambda(t)$$

that is $\lambda'\in L^1[t_0,T]$. But $\lambda(t)=0$ in $[0,t_0]$, we assert that λ' exists a. e. in $[0,T]$ and $\lambda'\in L^1[0,T]$ and

$$\lambda'_k \overset{w}{\rightharpoonup} \lambda' \quad \text{in} \quad L^1[0,T] \tag{2. 33}$$

In fact, since $\lim_{k\to\infty} t_k^*=t_0$, for any $\delta>0$ there is K^* such that for $k>K^*$,

$$t_0-\delta<t_k^*<t_0+\delta$$

From the definition of t_k^*, for any $g\in L^\infty[0,T]$, we have that when $k>K^*$

$$\begin{aligned}\left|\int_0^T(\lambda'_k-\lambda')g\right| &\leqslant \int_{t_0-\delta}^{t_0+\delta}|\lambda'_k-\lambda'||g|+\left|\int_{t_0+\delta}^{T}(\lambda'_k-\lambda')g\right| \\ &\leqslant \|g\|_{L^\infty}(|\lambda_k(t_0+\delta)-\lambda_k(t_0-\delta)| \\ &\quad +|\lambda(t_0+\delta)-\lambda(t_0-\delta)|)+\left|\int_{t_0+\delta}^{T}(\lambda'_k-\lambda')g\right|\end{aligned}$$

Let $k\to\infty$,

$$\overline{\lim_{k\to\infty}}\left|\int_0^T(\lambda'_k-\lambda')g\right|\leqslant 2\|g\|_{L^\infty}|\lambda(t_0+\delta)-\lambda(t_0-\delta)|\to 0 \quad (\delta\to 0)$$

Thus

$$\lim_{k\to\infty}\int_0^T\lambda'_k(t)g(t)dt=\int_0^T\lambda'(t)g(t)dt \qquad \forall\ g\in L^\infty[0,T]$$

i.e. (2. 33) holds. Similar to the proof of Theorem 2. 1, by (2. 30), (2. 31) and (2. 33) we can easily prove that (u,λ) is a solution of the problem (2. 1). The uniqueness is just a consequence of Theorem 2. 2 in [4].

(II) The proof of $n\geqslant 3$.

As $n=2$, suppose that $\psi_k=\psi-\varepsilon_k$. Let (u_k,λ_k) be the solution of (2. 1) corresponding to (φ,ψ_k) whose maximal existence-interval is $[0,t_{n,k}]$. From Theorem 2. 2 and the discussion in (I), we see that it remains to prove that $\{t_{n,k}\}$ has a positive lower bound.

From (2. 2) we have

$$\begin{aligned}\Psi_k(\lambda_k)(t)=&\int_0^x\frac{x-y}{x}(u_k(y,t)-\varphi(y))dy-\frac{1}{x}\int_0^t f(x+\lambda_k(\tau))u_k^m(x,\tau)d\tau \\ &+\frac{2}{x}\int_0^t\int_0^x f(y+\lambda_k(\tau))u_k^m(y,\tau)dyd\tau\end{aligned}$$

$$-\frac{1}{x}\int_0^t\int_0^x (x-y)f'(y+\lambda_k(\tau))u_k^m(y,\tau)dyd\tau$$
$$-\frac{1}{x}\int_0^t\int_0^x \lambda_k'(\tau)u_k(y,\tau)dyd\tau \qquad (\forall\, x>0)$$

where $\Psi_k(x)=-\int_0^x f(s)\psi_k(s)ds$ $(x<0)$. Thus there are positive constants C_1—C_5 depending only on K_1 (see the proof of Theorem 2.1) such that

$$|\Psi_k(\lambda_k)(t)|\leqslant (C_1+C_2t)x+C_3x^{\nu-1}t+C_4t/x+C_5t, \qquad \forall\, x>0$$

where $\nu=2(n-1)/(n-2)$. Therefore by $(H)_2'$ it follows that

$$|\lambda_k(t)|^{\alpha+1}\leqslant (C_1+C_2t)x+C_3x^{\nu-1}t+C_4t/x+C_5t, \quad \forall\, x>0$$

Choose that $x=(C_4t/(C_1+C_2t))^{1/2}$. Then

$$|\lambda_k(t)|\leqslant (2((C_1+C_2t)C_4t)^{1/2}+C_3(C_4/C_1)^{\nu-1}t^{(\nu+1)/2}+C_5t)^{1/(\alpha+1)}$$

This implies that there exists a constant T_* such that for $t\in[0,T_*]$

$$|\lambda_k(t)|\leqslant a_n=\frac{1}{n-2}$$

which shows that $t_{n,k}\geqslant T_*$, i.e. $\{t_{n,k}\}$ has a positive lower bound.

Remark 2.2 When $n\geqslant 3$, the condition $\psi(x)\leqslant -\varepsilon_0|x|^\alpha$ can be replaced by

$(H)_2^*$ $\quad \psi(x)\leqslant -\varepsilon_0(x+a_n)^\beta|x|^\alpha, \quad -a_n<x<0, \quad \beta,\alpha>-1.$

3. Existence and Uniqueness of (1.1)

In this section we study the existence and uniqueness of the problem (1.1). We began with:

Definition 3.1 *A function u and a surface Γ are said to be a solution of* (1.1) *if*

1) $u\geqslant 0, u\in C^0(\overline{G}_{\Gamma,T}\cap[\varepsilon,T])\cap L^\infty(\overline{G}_{\Gamma,T}), \quad \forall\, \varepsilon\in(0,T)$;

2) *Γ is a $W^{1,1}$ surface, that is, there exists a $W^{1,1}$ function $g(x,t)$ which determines Γ by $g(x,t)=0$;*

3) *For any $T'\in(0,T)$*

$$\iint_{G_{\Gamma,T'}}(uf_t+A(u)\Delta f)dxdt=-\int_{G_0}\varphi(x)f(x,0)dx-\iint_{\Gamma_{T'}}\psi(x)\nu_tfds$$

where $(\nu_x;\nu_t)$ is the normal to Γ, $f\in C^{2,1}(\overline{G}_{\Gamma,T'})$, $f(x,T')=0$ and $G_{\Gamma,T}$ is bounded by Γ, $\{t=0\}$ and $\{t=T\}$, $\Gamma_{T'}=\Gamma\cap(0,T')$.

The basic assumptions in this section are

$A(u)=u^m$, $m>1$ is a constant, $G_0=B_1(O)$ —— the unit ball and

$(H)_1'$: $\varphi(x)=\varphi(|x|)=\varphi(r)$ is bounded and measurable in $[0,T]$, $0\leqslant\varphi(r)\leqslant K_1$,

$(H)_n$: $\psi(x)=\psi(|x|)=\psi(r)$ is locally integrable in $[1,\infty)$ and there is a constant $C>0$ such that

if $n=2$, $\quad \psi(r)\leqslant -C(\ln r)^\beta, \quad 1<r<\infty, \beta>-1$;

if $n\geqslant 3$, $\quad \psi(r)\leqslant -C(r-1)^\beta r^\alpha, \quad 1<r<\infty, \beta>-1, \alpha<-n.$

Now suppose that φ,ψ satisfy $(H)_1'$ and $(H)_n$. Define φ_n,ψ_n as follows

if $n=2$, $\varphi_n(y)=\varphi(e^{-y})$, $y>0$, $\qquad \psi_n(y)=e^{-2y}\psi(e^{-y})$, $y<0$;

if $n\geqslant 3$, $\varphi_n(y)=\varphi(((n-2)y+1)^{-1/(n-2)})$, $\quad y>0$,

$$\psi_n(y)=((n-2)y+1)^{-2(n-1)/(n-2)}\cdot\psi(((n-2)y+1)^{1/(2-n)}),$$

$$-1/(n-2)<y<0 \tag{3.1}$$

Then (φ_n,ψ_n) satisfies $(H)'_1$ and $(H)'_2$ (or $(H)''_2$). In fact, for $n=2$, φ_n satisfies $(H)'_1$. Secondly, for $n=2$, by $(H)_n$ and (3.1),

$$\psi_2(y)=e^{-2y}\psi(e^{-y})\leqslant -Ce^{-2y}|y|^\beta\leqslant -C|y|^\beta \quad (\text{since} \quad y<0)$$

and $\beta>-1$. Thus ψ_n satisfies $(H)'_2$.

For $n\geqslant 3$ and $y\in\left(-\dfrac{1}{n-2},0\right)$,

$$\begin{aligned}\psi_n(y)&=((n-2)y+1)^{2(n-1)/(2-n)}\psi(((n-2)y+1)^{1/(2-n)})\\&\leqslant -C(1-((n-2)y+1)^{-1/(n-2)})^\beta\cdot((n-2)y+1)^{(\alpha+2(n-1))/(2-n)}\\&\leqslant -C'|y|^\beta\left(y+\frac{1}{n-2}\right)^{-(\alpha+2(n-1))/(n-2)}\end{aligned}$$

Since $\alpha<-n$, $-\dfrac{1}{n-2}(\alpha+2(n-1))>-\dfrac{1}{n-2}(-n+2n-2)=-1$. Also $\beta>-1$, thus ψ_n satisfies $(H)''_2$.

By Theorem 2.3 and Remark 2.2, the problem (2.1) has a unique solution $(v_n,\bar{\lambda}_n)$ corresponding to (φ_n,ψ_n), whose maximal existence-interval is $(0,T_n)$ and $\bar{\lambda}_n(t)>-a_n$ where $a_2=\infty$, and $a_n=1/(n-2)$ for $n\geqslant 3$. Define

$$\begin{cases}\lambda_n(t)=e^{-\bar{\lambda}_n(t)}, \quad u_n(x,t)=v_n(-\ln|x|-\bar{\lambda}_n(t),t), \quad n=2\\ \lambda_n(t)=((n-2)\bar{\lambda}_n(t)+1)^{1/(2-n)}\\ u_n(x,t)=v_n\left(\dfrac{|x|^{2-n}-1}{n-2}-\bar{\lambda}_n(t),t\right)\\ \Gamma_n \quad \text{is the surface} \quad |x|=\lambda_n(t), \quad n\geqslant 3\end{cases} \tag{3.2}$$

Then we have

Theorem 3.1 *Suppose that $A(u)=u^m$ $(m>1)$, $G_0=B_1(O)$ and φ,ψ satisfy $(H)'_1$ and $(H)_n$. Then (u_n,Γ_n) defined in (3.2) is a solution of the problem (1.1) corresponding to (φ,ψ).*

Proof We prove the theorem only for $n\geqslant 3$. For $n=2$ the proof is the same. For convenience, we omit the foot mark n and appoint k to denote the foot mark of sequence and n the dimension of Euclidean space.

First, we, in addition, assume that $\varphi(x)=\varphi(r)$ satisfies

(A): $\quad \varphi(r)\in C^1[0,1]$ and there is a constant $K_2>0$ such that

$$|(\varphi^m)'(r)|\leqslant K_2r^{1-n}, \quad r\in(0,1]$$

It is easily checked that $\varphi_n(y)$ (see (3.1)) satisfies $(H)_1$.

From Proposition 2.1, Theorems 2.1, 2.2 and Remark 2.1 we see that $v_x^m\in C^0(\overline{Q}_{0,T})\cap L^\infty(\overline{Q}_{0,T})$, $\bar{\lambda}\in W^{1,\infty}[0,T]$.

Assume that $\{\bar{\lambda}_k\}$ in $(P)_k$ is non-increasing,

$$\|\bar{\lambda}_k-\bar{\lambda}\|_{W^{1,1}}\to 0, \ \|\overline{\varphi}_k-\varphi_n\|_{L^1_{loc}[0,\infty)}\to 0 \text{ and } \varepsilon_k\to 0$$

From Section 2, we know that v_k converges to $v=v_\psi^\varphi$. Set

$$u_k(x,t)=v_k\left(\frac{|x|^{2-n}-1}{n-2}-\bar{\lambda}_k(t),t\right)$$

$$\lambda_k(t) = ((n-2)\bar{\lambda}_k(t) + 1)^{1/(2-n)}$$
$$\varphi_k(x) = \bar{\varphi}_k(((n-2)|x| + 1)^{1/(2-n)})$$
$$P_k(t) = ((n-2)(k + \bar{\lambda}_k(t) + 1))^{1/(2-n)}$$
$$G_k = \{(x,t); P_k(t) < |x| < \lambda_k(t), \quad 0 < t < T\}$$

Then u_k satisfies

$$\begin{cases} u_{kt} = \Delta u_k^m, & \text{in } G_k \\ u_k(x,0) = \varphi_k(x), & P_k(0) < |x| < 1 \\ u_k(x,t)|_{|x|=\lambda_k(t)} = \varepsilon_k, & 0 < t < T \\ u_k(x,t)|_{|x|=P_k(t)} = \varphi_k(k), & 0 < t < T \end{cases} \tag{3.3}$$

Let $0<\varepsilon<1$ and $G_{k,\varepsilon}=\{(x,t);\varepsilon<|x|<\lambda_k(t), 0<t<T\}$. Since $\lim\limits_{k\to\infty}P_k(t)=0$ uniformly, we may suppose that $G_{k,\varepsilon}\subset G_k, k=1,2,\cdots$. Put

$$G_T = \{(x,t); |x| < ((n-2)\lambda(t) + 1)^{1/(2-n)}, \quad 0 < t < T\}$$

Then $G_k\subset G_T$ because $\bar{\lambda}_k$ is decreasing. Therefore, for any $f\in C^{2,1}(\overline{G}_{T'}), f(x,T')=0$, where $T'\in(0,T)$, it follows, from (3.3), that

$$\iint_{G_{k,\varepsilon}} u_{kt} f dx dt = \iint_{G_{k,\varepsilon}} \Delta u_k^m f dx dt \tag{3.4}$$

By means of Green's formula

$$\iint_{G_{k,\varepsilon}} u_{kt} f dx dt = -\iint_{G_{k,\varepsilon}} u_k f_t dx dt + \iint_{\partial G_{k,\varepsilon}} u_k f v_t ds$$
$$= -\iint_{G_{k,\varepsilon}} u_k f_t dx dt - \int_{B_1(O)\setminus B_\varepsilon(O)} \varphi_k(x) f(x,0) dx$$

$$\iint_{G_{k,\varepsilon}} \Delta u_k^m f dx dt = \iint_{G_{k,\varepsilon}} u_k^m \Delta f dx dt + \int_0^{T'}\int_{|x|=\lambda_k(t)} \left(f\frac{\partial u_k^m}{\partial v} - u_k^m \frac{\partial f}{\partial v}\right) ds dt$$
$$- \int_0^{T'}\int_{|x|=\varepsilon} \left(f\frac{\partial u_k^m}{\partial v} - u_k^m \frac{\partial f}{\partial v}\right) ds dt \tag{3.5}$$

Obviously,

$$\lim_{k\to\infty}\iint_{G_{k,\varepsilon}} u_{kt} f dx dt = -\iint_{G_{\infty,\varepsilon}} u f_t dx dt - \int_{B_1(O)\setminus B_\varepsilon(O)} \varphi(x) f(x,0) dx \tag{3.6}$$

$$\lim_{k\to\infty}\iint_{G_{k,\varepsilon}} u_k^m \Delta f dx dt = \iint_{G_{\infty,\varepsilon}} u^m \Delta f dx dt \tag{3.7}$$

where $G_{\infty,\varepsilon}=G_{T'}\setminus\{B_\varepsilon(O)\times(0,T')\}$ and $u=u_n$ is defined by (3.2).

Now we prove

Lemma 3.1

$$\lim_{k\to\infty}\int_0^{T'}\int_{|x|=\lambda_k(t)}\left(f\frac{\partial u_k^m}{\partial v} - u_k^m\frac{\partial f}{\partial v}\right)ds dt = \int_0^{T'}\int_{|x|=\lambda_n(t)} f(x,t)\psi(\lambda_n(t))\lambda_n'(t) ds dt$$

where λ_n is defined by (3.2).

Proof By (3.3), we have

$$I_1 = \int_0^{T'} \int_{|x|=\lambda_k(t)} u_k^m \frac{\partial f}{\partial \nu} ds dt = \varepsilon_k^m \int_0^{T'} \int_{|x|=\lambda_k(t)} \frac{\partial f}{\partial \nu} ds dt$$

Then $I_1 \to 0 (k \to \infty)$ since $\varepsilon_k \to 0$ and $f \in C^{2,1}$ is bounded.

From the definition of u_k and $(P)_k$ one has

$$I_2 = \int_0^{T'} \int_{|x|=\lambda_k(t)} \frac{\partial u_k^m}{\partial \nu} f ds dt = \int_0^{T'} \int_{|x|=\lambda_k(t)} f \frac{\partial u^m}{\partial x_i} \frac{x_i}{|x|} ds dt$$

$$= -\int_0^{T'} \int_{|x|=\lambda_k(t)} f(x,t) v_{ky}^m(0,t) |x|^{1-n} ds dt$$

$$= -\int_0^{T'} v_{ky}^m(0,t) dt \int_{|x|=\lambda_k(t)} (\lambda_k(t))^{1-n} f(x,t) ds$$

$$= -\int_0^{T'} v_{ky}^m(0,t) \int_{|x|=\lambda_n(t)} (\lambda_n(t))^{1-n} f(x,t) ds dt + I_{22}$$

$$\equiv I_{21} + I_{22} \tag{3.8}$$

Set

$$\bar{f}(t) = \int_{|x|=\lambda_n(t)} (\lambda_n(t))^{1-n} f(x,t) ds$$

then

$$I_{21} = -\int_0^{T'} v_{ky}^m(0,t) \bar{f}(t) dt$$

Note $\bar{f} \in W^{1,1}[0,T]$. Similar to the proof of Theorem 2. 2 in [4] we get

$$\lim_{k\to\infty} I_{21} = -\int_0^{T'} v_y^m(0,t) \bar{f}(t) dt = -\int_0^{T'} \psi_n(\bar{\lambda}_n(t)) \bar{\lambda}_n'(t) \bar{f}(t) dt$$

(see also Proposition 2. 5) where ψ_n is given by (3. 1). Thus from (3. 1), we have

$$\lim_{k\to\infty} I_{21} = \int_0^{T'} \psi(\lambda_n(t)) (\lambda_n)^{n-1} \lambda_n'(t) \bar{f}(t) dt$$

$$= \int_0^{T'} \psi(\lambda_n(t)) (\lambda_n)^{n-1} \lambda_n'(t) \int_{|x|=\lambda_n(t)} (\lambda_n)^{1-n} f(x,t) ds dt$$

$$= \int_0^{T'} \int_{|x|=\lambda_n(t)} \psi(\lambda_n(t)) \lambda_n'(t) f(x,t) ds dt \tag{3.9}$$

Since φ satisfies (A), there exists a constant C independent of k such that $|v_{ky}^m(0,t)| \leqslant C$. Thus

$$|I_{22}| \leqslant C_1 \int_0^{T'} \left| \int_{|x|=\lambda_n(t)} (\lambda_n(t))^{1-n} f(x,t) ds - \int_{|x|=\lambda_k(t)} (\lambda_k(t))^{1-n} f(x,t) ds \right| dt$$

Also $T' < T$ and λ_k converges to λ_n uniformly. We have, therefore, that $\lim_{k\to\infty} I_{22} = 0$. This and (3. 8), (3. 9) yield

$$\lim_{k\to\infty} I_2 = \int_0^{T'}\int_{|x|=\lambda_n(t)} \psi(\lambda_n(t))\lambda_n'(t)f(x,t)dsdt$$

But $I_1 \to 0$, the proof is then complete.

Lemma 3. 2

$$\lim_{k\to\infty}\int_0^{T'}\int_{|x|=\varepsilon}\left(f\frac{\partial u_k^m}{\partial \nu} - u_k^m\frac{\partial f}{\partial \nu}\right)dsdt = \int_0^{T'}\int_{|x|=\varepsilon}\left(f\frac{\partial u^m}{\partial \nu} - u^m\frac{\partial f}{\partial \nu}\right)dsdt$$

Proof Obviously, it is only to prove

$$\lim_{k\to\infty}\int_0^{T'}\int_{|x|=\varepsilon} f\frac{\partial u_k^m}{\partial \nu}dsdt = \int_0^{T'}\int_{|x|=\varepsilon} f\frac{\partial u^m}{\partial \nu}dsdt$$

In fact,

$$\int_0^{T'}\int_{|x|=\varepsilon} f\frac{\partial u_k^m}{\partial \nu}dsdt = \int_0^{T'} v_{ky}^m\left(\frac{\varepsilon^{2-n}-1}{n-2} - \bar{\lambda}_k(t), t\right)\left(\varepsilon^{1-n}\int_{|x|=\varepsilon} f(x,t)ds\right)dt$$

$$= \int_0^{T'} v_{ky}^m\left(\frac{\varepsilon^{2-n}-1}{n-2} - \bar{\lambda}_k(t), t\right)g_1(\varepsilon,t)dt$$

where $g_1(\varepsilon,t) = \varepsilon^{1-n}\int_{|x|=\varepsilon} f(x,s)ds$. Define $g(0,t) = f(0,t)$. Then $g_1 \in C^0[0,T'] \cap C^1(0,T']$. Hence as it was done in the proof of (2. 40), one gets

$$\lim_{k\to\infty}\int_0^{T'} v_{ky}^m\left(\frac{\varepsilon^{2-n}-1}{n-2} - \bar{\lambda}_k(t), t\right)g_1(\varepsilon,t)dt$$

$$= \int_0^{T'} v_y^m\left(\frac{\varepsilon^{2-n}-1}{n-2} - \bar{\lambda}_n(t), t\right)g_1(\varepsilon,t)dt$$

$$= \int_0^{T'}\int_{|x|=\varepsilon} f\frac{\partial u^m}{\partial \nu}dsdt$$

which shows the lemma is valid.

We continue the proof of our theorem. From (3. 4)—(3. 7) and Lemmas 3. 1—3. 2, for any $\varepsilon \in (0,1)$ we have

$$\iint_{G_{\infty,\varepsilon}} (uf_t + u^m\Delta f)dxdt = -\int_{B_1(O)\setminus B_\varepsilon(O)} \varphi(x)f(x,0)dx$$

$$+\int_0^{T'}\int_{|x|=\lambda_n(t)} f(x,t)\psi(\lambda_n(t))\lambda_n'(t)dsdt - \int_0^{T'}\int_{|x|=\varepsilon}\frac{\partial u^m}{\partial \nu}f(x,t)dsdt \qquad (3. 10)$$

Notice

$$\int_0^{T'}\int_{|x|=\varepsilon}\frac{\partial u^m}{\partial \nu}fdsdt = \varepsilon^{1-n}\int_0^{T'} v_y^m\left(\frac{\varepsilon^{2-n}-1}{n-2} - \bar{\lambda}_n(t), t\right)\int_{|x|=\varepsilon} f(x,t)dsdt$$

$$= \delta^{(n-1)/(n-2)}\int_0^{T'} v_y^m\left(\frac{\delta-1}{n-2} - \bar{\lambda}_n(t), t\right)g_2(\delta,t)dt$$

where $\delta = \varepsilon^{2-n}$, $g_2(\delta,t) = \int_{|x|=\varepsilon} f(x,s)ds \in L^\infty$. Thus Proposition 2. 6 or Remark 2. 1 yields that for any $\sigma \in (0,T')$,

$$\lim_{\varepsilon\to 0}\varepsilon^{1-n}\int_\sigma^{T'} v_y^m\left(\frac{\varepsilon^{2-n}-1}{n-2}-\bar{\lambda}_n(t),t\right)g_2(\delta,t)dt$$

$$=\lim_{\delta\to\infty}\delta^{(n-1)/(n-2)}\int_\sigma^{T'} v_y^m\left(\frac{\delta-1}{n-2}-\bar{\lambda}_n(t),t\right)g_2(\delta,t)dt=0 \qquad (3.11)$$

On the other hand,

$$\left|\varepsilon^{1-n}\int_0^\sigma v_y^m\left(\frac{\varepsilon^{2-n}-1}{n-2}-\bar{\lambda}_n(t),t\right)\int_{|x|=\varepsilon} f(x,t)dsdt\right|$$

$$\leqslant \int_0^\sigma\left|v_y^m\left(\frac{\varepsilon^{2-n}-1}{n-2}-\bar{\lambda}_n(t),t\right)\right|\varepsilon^{1-n}\|f\|_{L^\infty}\left(\int_{|x|=\varepsilon} ds\right)dt$$

$$\leqslant C_n\|f\|_{L^\infty}\|v_y^m\|_{L^\infty}\cdot\sigma\leqslant 2C_n\|f\|_{L^\infty}\|\varphi\|_{C^1}\cdot\sigma\to 0 \qquad (\sigma\to 0)$$

Therefore letting $\varepsilon\to 0$ in (3.10) and using (3.11) and the inequality above, we obtain

$$\iint_{G_\infty}(uf_t+u^m\Delta f)dxdt=-\int_{B_1(O)}\varphi(x)f(x,0)dx+\int_0^{T'}\int_{|x|=\lambda_n(t)} f(x,t)\psi(\lambda_n)\lambda_n' dsdt \qquad (3.12)$$

Note that Γ is just the surface $|x|-\lambda_n(t)=0$. Thus $v_t=-\lambda_n'(t)$. This shows that (u_n,Γ) is a solution of (1.1). In other words, we have proved the theorem when (φ,ψ) satisfy (A) and $(H)_n$.

Now suppose that φ,ψ satisfy $(H)_1'$ and $(H)_n$. Choose φ_k such that they satisfy $(H)_1'$ and (A), $\{\varphi_k\}$ is decreasing and $\varphi_k\to\varphi$ in $L^1(B_1(O))$. Then by the known result, there is (u_k,λ_k) satisfying (3.11). Set

$$\lambda_k(t)=((n-2)\lambda_k^*(t)+1)^{1/(2-n)},\ u_k(x,t)=v_k\left(\frac{|x|^{2-n}-1}{n-2}-\lambda_k^*(t),t\right)$$

Then (v_k,λ_k^*) is a solution of (2.1) corresponding to

$$\varphi_k^*(x)=\varphi_k(((n-2)x+1)^{1/(2-n)})$$

$$\psi_k^*(x)=((n-2)x+1)^{-2(n-1)/(n-2)}\psi(((n-2)x+1)^{1/(2-n)})$$

Notice that $\{\varphi_k^*\}$ is decreasing. From Proposition 2.2 it is seen that $\{v_k\}$, and $\{u_k\}$, is decreasing, $\{\lambda_k^*\}$ increasing, i.e., $\{\lambda_k\}$ is decreasing. As it was done in the proofs of Theorems 2.2, 2.3, we may prove that $\{T_k\}$ has positive bound and therefore $T_0\equiv\lim T_k>0$. Meanwhile, there exist $\lambda^*\in W^{1,1}[0,T]$ with $(\lambda^*)'\leqslant 0,\lambda^*(0)=0$, and $v\in C^0(\bar{Q}_{0,T_0}\cap[\varepsilon,T_0])\cap L^\infty(\bar{Q}_{0,T_0})$ such that

$$\|(\lambda_k^*)'-(\lambda^*)'\|_{L^1_{weak}[0,T]}\to 0,\quad \|v_k-v\|_{C^0_{loc}(\bar{Q}_{0,T}\cap[\varepsilon,T_0])}\to 0 \quad (k\to\infty) \qquad (3.13)$$

Hence $\lim_{k\to\infty}\lambda_k(t)=((n-2)\lambda^*(t)+1)^{1/(2-n)}\equiv\lambda(t)$ uniformly in $[0,T_0]$ and

$$\|u_k-u\|_{C^0_{loc}(\bar{G}_{\lambda,T_0}\cap[\varepsilon,T_0))}\to 0$$

where $G_{\lambda,T_0}=\{(x,t);|x|<\lambda(t),0<t<T_0\}$ and $u(x,t)=v\left(\frac{|x|^{2-n}-1}{n-2}-\lambda(t),t\right)$. Using (3.13) once again, one gets that $\lambda_k'\xrightarrow{w}\lambda'$ in $L^1[0,T'],\forall\ T'\in(0,T_0)$.

Without loss of generality, we assume $T'<T_k$ for all k. Since (u_k,λ_k) satisfies

(3.12), we have, for any $T'<T_0$, that

$$\iint_{G_{\lambda_k,T'}} (u_k f_t + u_k^m \Delta f) dx dt$$

$$= -\int_{B_1(O)} \varphi_k(x) f(x,0) dx + \int_0^{T'} \int_{|x|=\lambda_k(t)} f(x,t)\psi(\lambda_k(t))\lambda_k'(t) ds dt$$

where $f\in C^{2,1}$ and $f(x,T')=0$.

Clearly,

$$\lim_{k\to\infty}\int_{B_1(O)} \varphi_k(x) f(x,0) dx = \int_{B_1(O)} \varphi(x) f(x,0) dx \tag{3.14}$$

$$\lim_{k\to\infty}\iint_{G_{\lambda_k,T'}} (u_k f_t + u_k^m \Delta f) dx dt = \iint_{G_{\lambda,T}} \varphi(x) f(x,0) dx \tag{3.15}$$

But

$$\left|\int_0^{T'}\int_{|x|=\lambda_k(t)} f(x,t)\psi(\lambda_k(t))\lambda_k'(t) ds dt - \int_0^{T'}\int_{|x|=\lambda(t)} f(x,t)\psi(\lambda(t))\lambda'(t) ds dt\right|$$

$$\leqslant \left|\int_0^{T'}\psi(\lambda_k(t))\lambda_k'(t)\left[\int_{|x|=\lambda_k(t)} f(x,t) ds - \int_{|x|=\lambda(t)} f(x,t) ds\right] dt\right|$$

$$+ \left|\int_0^{T'}\int_{|x|=\lambda(t)} f(x,t)[\psi(\lambda_k(t))\lambda_k'(t) - \psi(\lambda(t))\lambda'(t)] ds dt\right|$$

$$\equiv J_1 + J_{J_2} \tag{3.16}$$

Since λ_k converges to λ uniformly, one gets

$$\lim_{k\to\infty}\sup_{[0,T']}\left|\int_{|x|=\lambda_k(t)} f(x,t) ds - \int_{|x|=\lambda(t)} f(x,t) ds\right| = 0$$

Thus $\lim_{k\to\infty} J_1 = 0$.

To prove $\lim_{k\to\infty} J_2 = 0$, we first notice that

$$|\psi(\lambda_k(t))\lambda_k'(t)| = |\psi_k^*(\lambda_k^*(t))((n-2)\lambda_k^*(t)+1)^{(n-1)/(n-2)}(\lambda_k^*)'(t)|$$

$$\leqslant |\psi_k^*(\lambda_k^*(t))(\lambda_k^*)'(t)| = |v_{kr}(0,t)| \leqslant C(\sigma), \quad \text{a. e. in} \quad [\sigma, T']$$

see Propositions 2.4, 2.5. And $\psi(\lambda_k(t))\lambda_k'(t)$ is also non-positive we obtain that

$$\psi(\lambda_k(t))\lambda_k'(t) \xrightarrow{w} \psi(\lambda(t))\lambda'(t) \quad \text{in} \quad L^1[\sigma, T']$$

Therefore for any $\sigma\in(0,T')$,

$$\overline{\lim_{k\to\infty}}|J_2| = \overline{\lim_{k\to\infty}}\left|\int_\sigma^{T'}(\psi(\lambda_k(t))\lambda_k'(t) - \psi(\lambda(t))\lambda'(t))\left(\int_{|x|=\lambda(t)} f(x,t) ds\right) dt\right|$$

$$+ \overline{\lim_{k\to\infty}}\left|\int_0^\sigma(\psi(\lambda_k(t))\lambda_k'(t) - \psi(\lambda(t))\lambda'(t))\left(\int_{|x|=\lambda(t)} f(x,t) ds\right) dt\right|$$

$$\leqslant \overline{\lim_{k\to\infty}} \| f \|_{L^\infty}\omega_n|\lambda(T')|\left(\left|\int_0^\sigma \psi(\lambda_k(t))\lambda_k'(t) dt\right| + \left|\int_0^\sigma \psi(\lambda(t))\lambda'(t) dt\right|\right)$$

$$\leqslant 2\omega_n \| f \|_{L^\infty}|\lambda(T')|\left|\int_1^{\lambda(\sigma)}|\psi(s)| ds \to 0 \quad (\sigma\to 0)\right.$$

where ω_n is the measure of unit sphere in R^n. Thus $\lim_{k\to\infty} J_2=0$. It then follows, from (3.16), that

$$\lim_{k\to\infty}\int_0^{T'}\int_{|x|=\lambda_k(t)} f(x,t)\psi(\lambda_k(t))\lambda_k'(t)dsdt=\int_0^{T'}\int_{|x|=\lambda(t)} f(x,t)\psi(\lambda(t))\lambda'(t)dsdt$$

This and (3.14), (3.15) yield that

$$\iint_{G_{\lambda,T'}}(uf_t+u^m\Delta f)dxdt=-\int_{B_1(O)}\varphi(x)f(x,0)dx+\int_0^{T'}\int_{|x|=\lambda(t)} f(x,t)\psi(\lambda(t))\lambda'(t)dsdt$$

where $f\in C^{2,1}(\overline{G}_{\lambda,T'})$, $f(x,T')=0$. Obviously, u and Γ: $|x|-\lambda(t)=0$ also satisfy the other conditions in Definition 3. 1. Thus (u,Γ) is a solution of the problem (1. 1).

The uniqueness of solutions of the problem (1. 1) is based on the follwing proposition

Proposition 3. 1 (Brézis and Crandall [1]) *Suppose that $A(u)$ satisfies*

$(H)_{BC}$: $A: R^1\to R^1$ is non-decreasing and continuous, $A(0)=0$.

Let u and $\hat{u}$ satisfy $u-\hat{u}\in L^\infty(Q)\cap L^1(Q)$, $A(u)-A(\hat{u})\in L^\infty(Q)$, where $Q=R^n\times(0,T)$. If for any $f\in C_0^\infty(R^n\times[0,T))$ there holds

$$\int_0^T\int_{R^n}((u-\hat{u})f_t+(A(u)-A(\hat{u}))\Delta f)dxdt=0$$

then $u=\hat{u}$ a. e..

This proposition is a summary survey of Theorem 1, Proposition 1 and Remark (1. 22) in [1].

Theorem 3. 2 *Under the hypotheses in Theorem 3. 1, the solution of (1. 1) is unique.*

Proof Suppose that (u,Γ) is the solution obtained in Theorem 3. 1. Then there exists a $W^{1,1}$ function $\lambda(t)$ such that Γ is determined by $|x|-\lambda(t)=0$. What we want to prove is that if (v,Γ') is another solution of (1. 1) then $u=v$, $\Gamma=\Gamma'$.

In fact, since (v,Γ') satisfies

$$\iint_{G_{\Gamma',T'}}(vf_t+v^m\Delta f)dxdt=-\int_{B_1(O)}\varphi(x)f(x,0)dx+\iint_{\Gamma'_{T'}} f(x,t)\psi(x)v_t dsdt \tag{3.17}$$

where $\Gamma'_{T'}=\Gamma\cap[0,T']$, $f\in C^{2,1}(\overline{G}_{\Gamma'})$, $f(x,T')=0$.

On the other hand, denote by $B_R(T')$ the set $\{(x,t); |x|<R, 0<t<T'\}$. Choose R such that $B_R(T')\supset G_{\Gamma',T'}$. For any $f\in C^{2,1}(B_R(T'))$, $f|_{|x|=R}=f|_{t=T'}=0$, (3. 17) holds. Moreover,

$$\iint_{B_R(T')\setminus G_{\Gamma',T'}}\psi(x)f_t dxdt=-\int_{B_R(O)\setminus B_1(O)}\psi(x)f(x,0)dx+\iint_{\Gamma'} f(x,t)\psi(x)v_t ds \tag{3.18}$$

Thus if we set

$$V(x,t)=\begin{cases}v(x,t) & \text{in } G_{\Gamma',T'}\\ \psi(x) & \text{in } B_R(T')\setminus G_{\Gamma',T'}\end{cases}$$

$$A(u)=\begin{cases}u^m & u\geqslant 0\\ 0 & u<0\end{cases}$$

then for $f\in C^{\infty}(R^n\times(0,T'))$, $f(x,T')=0$, $f(x,t)=0$, for $|x|$ large enough, we have

$$\int_0^{T'}\int_{R^n}(Vf_t+A(V)\Delta f)dxdt=-\int_{R^n}\phi(x)f(x,0)dx$$

where $\phi(x)=\varphi(x)$, $x\in B_1(O)$, $\phi(x)=\psi(x)$, $x\notin B_1(O)$.

Similarly, setting

$$U(x,t)=\begin{cases}u(x,t) & \text{in } G_{\Gamma}\\ \psi(x) & \text{otherwise}\end{cases}$$

we also have

$$\int_0^{T'}\int_{R^n}(Uf_t+A(U)\Delta f)dxdt=-\int_{R^n}\phi(x)f(x,0)dx$$

Thus for any $f\in C_0^{\infty}(R^n\times(0,T))$ one has

$$\int_0^{T}\int_{R^n}((U-V)f_t+(A(U)-A(V))\Delta f)dxdt=0$$

Clearly, $A=u^m$ satisfies $(H)_{BC}$ in Proposition 3. 1. Notice for $T'<T$, $G_{\Gamma'}$ and G_{Γ} are all bounded domains. Therefore for $|x|$ large enough, $U-V=0$. Thus $U-V\in L^1\cap L^{\infty}$. In the same way, we have $A(U)-A(V)\in L^{\infty}$. Hence by Proposition 3. 1, we conclude that $U=V$. This and the definitions of U and V and $\psi<0$ yield $u=v$, thus $\Gamma'=\Gamma$.

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References

[1] Brezis H. & Crandall M. G., Uniqueness of solutions of the initial value problem for $u_t=\Delta\varphi(u)$, *J. Math. pures et appl.*, **58**(1979), 153—163.

[2] Friedman A., Partial Differential Equations of Parabolic Type, Prentice-Hall, Englewood Cliffs, N. J., 1964.

[3] Gilding B. H., Hölder continuity of solutions of parabolic equations, *J. London Math. Soc.*, **12** (1976).

[4] Li Huilai, Free boundary problems for degenerate parabolic equations, *Lecture Notes in Math.*, **1306**, 102—130.

[5] Kruzhkov S. N., Results concerning the nature of the continuity of solutions of parabolic equations and some of their applications, *Matematicheskie, Zametki* Vol. **6**, No. 1—2(1969)(97—108), 517—532.

[6] Ladyzenskaja O. A. et al., Linear and Quasilinear Equations of Parabolic Type, Amer. Math. Soc. Transl. R. J., 1968.

[7] Wu Zhuoqun, A free boundary problem for degenerate quasilinear parabolic equations., *Nonlinear Ana. Theory, Methods and Appl.* Vol. **9**, No. **9**(1985), 937—951.

[8] Kalashnikov A. S., The propagation of disturbance in problems of nonlinear heat condition with absorption, *Zh. Vychisl. Mat. mat. Fiz.*, **144**(1971)(890—907), 70—85.