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ASYMPTOTIC BEHAVIOR OF SOLUTIONS FOR THE BELOUSOV-ZHABOTINSKII REACTION DIFFUSION SYSTEM

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Abstract

The present paper characterizes the asymptotic behavior of the timedependent solution of the coupled Belousov-Zhabotinskii reaction diffusion equations in relation to the steady-state solutions of the corresponding boundary value problem. This characterization leads to an explicit relationship among the various physical constants and the boundary and initial functions.

1. Introduction

In some chemical reaction problems, a simplified model for the concentration densities $u \equiv u(t, x)$, $v = v(t, x)$ of two reactants, such as bromous acid and bromide ion, is given by a coupled system of reaction diffusion equations in the form

$$\begin{aligned} u_t - D_1 \nabla^2 u &= u(a - bu - cv) \\ v_t - D_2 \nabla^2 v &= -c_1 uv \end{aligned} \qquad (t > 0,\ x \in \Omega) \tag{1.1}$$

Where D_1, D_2, a, b, c and c_1 are positive constants and Ω is the reaction-diffusion medium. The coupled system is often referred to as the Belousov-Zhabotinskii chemical reaction equations and has been given considerable attention in recent years (cf. [1-5, 10]). Much discussion of Eq. (1.1) is devoted to the traveling wave solution in the one-dimensional spatial domain $\Omega \equiv R^1$. When Ω is a general bounded domain in R^n, Eq. (1.1) is supplemented by a boundary condition in the form

$$\begin{aligned} \alpha(x)\, \partial u/\partial v + \beta(x) u &= 0 \\ \alpha(x)\, \partial v/\partial v + \beta(x) v &= 0 \end{aligned} \qquad (t > 0,\ x \in \partial\Omega) \tag{1.2}$$

together with the initial condition

$$u(0, x) = u_0(x), \quad v(0, x) = v_0(x) \qquad (x \in \Omega) \tag{1.3}$$

where $\alpha \geq 0$, $\beta \geq 0$ with $\alpha + \beta > 0$, $\partial/\partial v$ is the outward normal derivative on $\partial\Omega$, and $u_0 \geq 0$, $v_0 \geq 0$ in Ω. It is assumed that the functions in (1.2) (1.3) and the domain Ω are smooth and $\beta(x)$ is not identically zero (see [5] for the case $\beta(x) \equiv 0$).

It has been shown in [5] that for any nonnegative initial function u_0, v_0, problem (1.1) - (1.3) has a unique nonnegative solution (u, v). The aim of this paper is to give a more precise description about the asymptotic behavior of the solution (u, v) in relation to the steady-state solutions of the corresponding boundary-value problem

$$\begin{aligned} -D_1 \nabla^2 u &= u(a - bu - cv) \\ -D_2 \nabla^2 v &= -c_1 uv \end{aligned} \qquad (t > 0,\ x \in \Omega) \tag{1.4}$$

$$\begin{aligned} \alpha(x)\, \partial u/\partial v + \beta(x) u &= 0 \\ \alpha(x)\, \partial v/\partial v + \beta(x) v &= 0 \end{aligned} \qquad (t > 0,\ x \in \partial\Omega) \tag{1.5}$$

Since problem (1.4) (1.5) has the trivial solution (0, 0) it is interesting to know when it has a nontrivial solution, and whether and when the time dependent solution (u, v) converges to the nontrivial solution. Our main results characterize the asymptotic behavior of the solution (u, v) in terms of the various physical constants in (1.1) as well as the effect of the boundary and initial conditions (1. 2) (1. 3).

2. The Main Results

The characterization of the existence of a nontrivial steady-state solution and its relation to the time-dependent solution is based on the smallest eigenvalue λ_0 and its corresponding eigenfunction $\varphi(x)$ of the eigenvalue problem

$$\nabla^2\varphi + \lambda\varphi = 0 \quad \text{in } \Omega, \ B[\varphi] = 0 \quad \text{on } \partial\Omega \tag{2. 1}$$

where $B[w] \equiv \alpha(x)\,\partial w/\partial\nabla + \beta(x)\,w$ for any function w. It is well-known that $\lambda_0 > 0$ and $\varphi(x) > 0$ in Ω. When $\alpha(x) > 0$ the maximum principle implies that $\varphi(x) > 0$ on $\bar{\Omega}$. We normalize φ so that $\max \varphi(x) = 1$ on $\bar{\Omega}$. The following existence result for the scalar boundary value problem

$$-D_1\nabla^2 U = U(a - bU) \ \text{ in } \Omega, \ B[U] = 0 \ \text{ on } \partial\Omega \tag{2. 2}$$

is well-known.

Lemma 2. 1. *Problem (2. 2) has only the trivial solution $U = 0$ when $a \le \lambda_0 D_1$; and it has a unique positive solution $U_s(x)$ when $a > \lambda_0 D_1$.*

A proof of the above lemma can be found in [9, p. 1174]. Based on the solution U_s of problem (2. 2) we state our main results in the following two theorems.

Theorem 1. *The steady-state problem (1. 4) (1. 5) has only the trivial solution (0, 0) when $a \le \lambda_0 D_1$; and it has exactly two solutions (0, 0) and (U_s, 0) when $a > \lambda_0 D_1$, where U_s is the unique positive solution of (2. 2).*

Theorem 2. *Let (u, v) be the nonnegative solution of (1. 1) - (1. 3) with any $(u_0, v_0) \ge (0, 0)$ and let U_s be the positive solution of problem (2. 2). Then*

$$\lim_{t\to\infty} (u(t, x), \ v(t, x)) = (0, \ 0) \tag{2. 3}$$

when $a \le \lambda_0 D_1$ or when $u_0(x) \equiv 0$, and

$$\lim_{t\to\infty} (u(t, x), \ v(t, x)) = (U_s(x), \ 0) \tag{2. 4}$$

when $a > \lambda_0 D_1$ and $u_0(x) \ge \varepsilon\varphi(x)$, where $\varepsilon > 0$ can be arbitrarily small.

Remark 2. 1. When $a > \lambda_0 D_1$ the conclusion in (2. 4) also hold for any $U_0(x) \not\equiv 0$ provided that $\alpha(x) > 0$. For in this situation the maximum principle implies that $u(t, x) > 0$ on $R^+ \times \bar{\Omega}$. By considering problem (1. 1) - (1. 3) with the initial functions $u(t_1, x)$, $v(t_1, x)$ in the domain $[t_1, \infty) \times \Omega$ for a fixed $t_1 > 0$, the requirement $u(t_1, x) \ge \varepsilon\varphi(x)$ for some $\varepsilon > 0$ is clearly satisfied. It follows from the uniqueness property of the solution (u, v) that (2. 4) holds.

3. Proof of the Main Theorems

Proof of Theorem 1. Let $(U_s(x), V_s(x))$ be any nonnegative solution of (1. 4) (1. 5). Multiplying both equations in (1. 4) by $\varphi(x)$ and integrating over Ω yield

$$-D_1\int_\Omega \varphi\nabla^2 U_s dx = \int_\Omega \varphi U_s(a - bU_s - cV_s)\,dx$$

$$-D_2\int_\Omega \varphi\nabla^2 V_s dx = -c_1\int_\Omega \varphi U_s V_s dx$$

By applying the Green's theorem and using the boundary condition (1. 5) the above equations become

$$\begin{aligned} (\lambda_0 D_1 - a)\int_\Omega \varphi U_s dx &= -\int_\Omega (b\varphi U_s^2 + c\varphi U_s V_s)\,dx \\ \lambda_0 D_2\int_\Omega \varphi V_s dx &= -c_1\int_\Omega \varphi U_s V_s dx \le 0 \end{aligned} \tag{3. 1}$$

Since $\varphi(x) > 0$ in Ω, the second relation in (3. 1) implies that $V_s(x) = 0$ in Ω.

Similarly the first relation yields $u_S(x) = 0$ when $a \le \lambda_0 D_1$. This shows that the only nonnegative solution to (1. 4) (1. 5) is the trivial solution (0, 0) when $a \le \lambda_0 D_1$. It also shows that when $a > \lambda_0 D_1$ the steady-state solution must be in the form $(U_S, 0)$ where U_S is a solution of (2. 2). But by Lemma 2. 1, U_S is the unique positive solution of (2. 2) when $a > \lambda_0 D_1$. We conclude that $(U_S, 0)$ is the only nontrivial solution of (1. 4) (1. 5). This proves the theorem.

In order to prove theorem 2 we prepare the following Lemmas for the case $a > \lambda_0 D_1$.

Lemma 3. 1. *Let $a > \lambda_0 D_1$ and let $(\bar{u}(t, x), \underline{v}(t, x))$ be the solution of (1. 1) - (1. 3) corresponding to $u_0 = M_1$, $v_0 = 0$, where M_1 is any constant satisfying $M_1 \ge a/b$. Then $\underline{v}(t, x) \equiv 0$ and $\bar{u}(t, x)$ converges monotonically from above to a solution $\bar{u}_S$ of (2. 2).*

Proof. It is easily seen that $\underline{v} \equiv 0$ and $\bar{u}$ is the solution of the scalar boundary value problem

$$\bar{u}_t - D_1 \nabla^2 \bar{u} = \bar{u}(a - b\bar{u}), \quad B[\bar{u}] = 0, \quad \bar{u}(0, x) = M_1 \tag{3. 2}$$

Since $\tilde{u} \equiv M_1$ and $\hat{u} \equiv 0$ are upper and lower solutions of (3. 2) the standard comparison theorem ensures that $0 \le \bar{u}(t, x) \le M_1$ (cf. [6, 8]). For fixed $h > 0$, define $w(t, x) = \bar{u}(t, x) - \bar{u}(t + h, x)$. Then

$$\begin{aligned} w_t - D_1 \nabla^2 w &= \bar{u}(t, x)(a - b\bar{u}(t, x)) - \bar{u}(t + h, x)(a - b\bar{u}(t + h, x)) \\ &= [a + b(\bar{u}(t, x) + \bar{u}(t + h, x))] w \end{aligned} \tag{3. 3}$$

Since $B[w] = 0$ and $w(0, x) = M_1 - \bar{u}(h, x) \ge 0$ the maximum principle implies that $w(t, x) \ge 0$. This shows that $\bar{u}(t, x)$ is monotone nonincreasing. It follows from $\bar{u} \ge 0$ that $\lim \bar{u}(t, x) = \bar{u}_S(x)$ as $t \to \infty$ exists. The same argument as in [8, 9] shows that $\bar{u}_S$ is a steady-state solution of (2. 2). This proves the lemma.

Lemma 3. 2. *Let δ_1, δ_2 be any positive constants such that $b\delta_1 + c\delta_2 \le a - \lambda_0 D_1$ and let $(\underline{u}(t, x), \bar{v}(t, x))$ be the solution of (1. 1) - (1. 3) corresponding to $u_0 = \delta_1 \varphi(x)$, $v_0 = \delta_2$ where $a > \lambda_0 D_1$. Then $\underline{u}(t, x)$ is monotone nondecreasing and $\bar{v}(t, x)$ is monotone nonincreasing in t. Moreover,*

$$\lim_{t \to \infty} (\underline{u}(t, x), \bar{v}(t, x)) = (U_S(x), 0) \tag{3. 4}$$

where U_S is the positive solution of (2. 2).

Proof. From $\underline{u} \ge 0$, the constants $\tilde{v} = \delta_2$ and $\hat{v} = 0$ are upper and lower solutions of the scalar boundary value problem

$$\bar{v}_t - D_2 \nabla^2 \bar{v} = -(c_1 \underline{u}) \bar{v}, \qquad B[\bar{v}] = 0, \qquad \bar{v}(0, x) = \delta_2 \tag{3. 5}$$

This implies that

$$0 \le \bar{v}(t, x) \le \delta_2 \qquad (t > 0, \ x \in \bar{\Omega}) \tag{3. 6}$$

Similarly from $\bar{v} \ge 0$ the pair $\tilde{u} = a/b$ and $\hat{u} = \delta_1 \varphi$ are upper and lower solutions of the scalar boundary value problem

$$\underline{u}_t - D_1 \nabla^2 \underline{u} = \underline{u}(a - b\underline{u} - c\bar{v}), \quad B[\underline{u}] = 0, \quad \underline{u}(0, x) = \delta_1 \varphi(x) \tag{3. 7}$$

if

$$-\delta D_1 \nabla^2 \varphi \le \delta \varphi (a - b\delta_1 \varphi - c\bar{v})$$

By the relation (2. 1) the above inequality is equivalent to $\lambda_0 D_1 \le a - b\delta_1 \varphi - c\bar{v}$. In view of (3. 6) and $\varphi \le 1$ this relation holds by the condition on δ_1 and δ_2. Since $\delta_1 \le a/b$ the comparison theorem for problem (3. 7) implies that

$$\delta_1 \varphi(x) \le \underline{u}(t, x) \le a/b \qquad (t > 0, \ x \in \bar{\Omega}) \tag{3. 8}$$

Now fix $h > 0$ and define

$$\underline{w}(t, x) = \underline{u}(t + h, x) - \underline{u}(t, x), \quad \bar{w}(t, x) = \bar{v}(t, x) - \bar{v}(t + h, x)$$

Then by (1. 1)

$$\underline{w}_t - D_1 \nabla^2 \underline{w} = \underline{u}(t + h, x)[a - b\underline{u}(t + h, x) - c\bar{v}(t + h, x)]$$

$$-\underline{u}(t,x)[a-b\underline{u}(t,x)-c\bar{v}(t,x)]$$
$$=[a-b(\underline{u}(t+h,x)+\underline{u}(t,x))-c\bar{v}(t+h,x)]\underline{w}(t,x)$$
$$+(c\underline{u}(t,x))\bar{w}(t,x)$$
$$\bar{w}_t-D_2\nabla^2\bar{w}=-c_1[\underline{u}(t,x)\bar{v}(t,x)-\underline{u}(t+h,x)\bar{v}(t+h,x)]$$
$$=-(c_1\underline{u}(t,x))\bar{w}(t,x)+(c_1\bar{v}(t+h,x))\underline{w}(t,x)$$

The above relation is equivalent to

$$\begin{aligned}\underline{w}_t-D_1\nabla^2\underline{w}+\sigma_1(t,x)\underline{w}&=(c\underline{u}(t,x))\bar{w}(t,x)\\ \bar{w}_t-D_2\nabla^2\bar{w}+\sigma_2(t,x)\bar{w}&=(c_1\bar{v}(t+h,x))\underline{w}\end{aligned} \tag{3. 9}$$

where

$$\begin{aligned}\sigma_1(t,x)&=a-b(\underline{u}(t+h,x)+\underline{u}(t,x))-c\bar{v}(t+h,x)\\ \sigma_2(t,x)&=c_1\underline{u}(t,x)\end{aligned} \tag{3. 10}$$

Since $B[\underline{w}]=B[\bar{w}]=0$ and by (3. 6) and (3. 8)

$$\underline{w}(0,x)=\underline{u}(h,x)-\delta_1\varphi(x)\geq 0,\quad \bar{w}(0,x)=\delta_2-\bar{v}(h,x)\geq 0$$

the maximum principle for weakly coupled system implies that $\underline{w}\geq 0$, $\bar{w}\geq 0$ (cf. [7], see also[5]). This shows that $\underline{u}(t,x)$ is nondecreasing and $\bar{v}(t,x)$ is nonincreasing in t. It follows from the bounded property of $\underline{u}$ and $\bar{v}$ that the limits

$$\lim_{t\to\infty}\underline{u}(t,x)=\underline{u}_S(x),\quad \lim_{t\to\infty}\bar{v}(t,x)=\bar{v}_S(x)$$

exist and $\delta_1\varphi\leq\underline{u}_S\leq a/b$ and $0\leq\bar{v}_S(x)\leq\delta_2$. By the same argument as in [9], $(\underline{u}_S,\bar{v}_S)$ is a solution of problem (1. 4) (1. 5). Since $u_S\geq\delta_1\varphi$ and by Theorem 1. $(U_S,0)$ is the only nontrivial solution of (1. 4) (1. 5). We conclude that $(\underline{u}_S,\bar{v}_S)=(U_S,0)$. This proves the relation (3. 4).

Lemma 3. 3. *Let* $(\bar{u},\underline{v})$, $(\underline{u},\bar{v})$ *be the solutions given in Lemmas* 1 *and* 2. *Then*

$$\bar{u}\geq\underline{u},\ \bar{v}\geq 0.$$

Moreover

$$\begin{aligned}&\lim\bar{u}(t,x)=\lim\underline{u}(t,x)=U_S(x)\\ &\lim\bar{v}(t,x)=0\qquad\qquad as\ t\to\infty\end{aligned} \tag{3. 11}$$

Proof. Let $w=\bar{u}-\underline{u}$. Then w satisfies the differential relation (3. 3) except with $\bar{u}(t+h,x)$ replaced by $\underline{u}(t,x)$. Since $B[w]=0$ and $\bar{u}(0,x)-\underline{u}(0,x)=M_1-\delta_1\varphi(x)\geq 0$ the maximum principle implies that $w\geq 0$. This leads to the conclusion $\bar{u}\geq\underline{u}$. Moreover the results of Lemmas 1 and 2 ensure that $\lim\bar{u}(t,x)=\bar{u}_s(x)\geq\delta_1\varphi(x)$ as $t\to\infty$ and therefore $\bar{u}_S$ must coincide with $U_S(x)$. This proves the first relation in (3. 11). The result $\bar{v}\geq 0$ and the second limit in (3. 11) also follow from Lemmas 1 and 2.

Proof of Theorem 2. Let $U(t,x)$ and $V(t,x)$ be the solution of the respective scalar boundary value problems

$$U_t-D_1\nabla^2U=U(a-bU),\quad B[U]=0,\quad U(0,x)=u_0(x) \tag{3. 12}$$

$$V_t-D_2\nabla^2V=0\qquad B[V]=0,\quad V(0,x)=v_0(x) \tag{3. 13}$$

By the comparison theorem for scalar problems, the solution (u,v) of (1. 1) — (1. 3) satisfies the relation $0\leq u\leq U$, $0\leq v\leq V$. Since $\lim U(t,x)=\lim V(t,x)=0$ as $t\to\infty$ when $a\leq\lambda_0D_1$ the result in (2. 3) follows immediately. When $u_0\equiv 0$ the solution of (1. 1) — (1. 3) is in the form of $(0,v)$ and thus it always converges to $(0,0)$ for any $a>\lambda_0D_1$.

To show the relation (2. 4) when $a>\lambda_0D_1$ and $u_0(x)\not\equiv 0$ we observe from Lemma 3. 3 that it suffices to establish the relation $\underline{u}\leq u\leq\bar{u}$, $0\leq v\leq\bar{v}$ for the solution (u, v). By the comparison theorem in [5] this relation holds if $(\tilde{u}, \tilde{v}) = (\bar{u}, \bar{v})$ and $(\hat{u}, \hat{v}) = (\underline{u}, 0)$ are upper-lower solutions of the coupled system (1.1)-(1.3), that is, $(\tilde{u}, \tilde{v})$ and $(\hat{u}, \hat{v})$ satisfy the differential inequalities:

$$\begin{aligned} &\tilde{u}_t - D_1\nabla^2\tilde{u} \geq \tilde{u}(a - b\tilde{u} - c\hat{v}) \\ &\tilde{v}_t - D_2\nabla^2\tilde{v} \geq -c_1\hat{u}\tilde{v} \qquad (t>0,\ x\in\Omega) \\ &\hat{u}_t - D_1\nabla^2\hat{u} \leq \hat{u}(a - b\hat{u} - c\tilde{v}) \\ &\hat{v}_t - D_2\nabla^2\hat{v} \leq -c_1\tilde{u}\hat{v} \end{aligned} \tag{3.14}$$

the boundary inequalities

$$B[\tilde{u}] \geq 0 \geq B[\hat{u}],\quad B[\tilde{v}] \geq 0 \geq B[\hat{v}] \qquad (t>0,\ x\in\partial\Omega) \tag{3.15}$$

and the initial inequalities

$$\begin{aligned} &\tilde{u}(0, x) - u_0(x) \geq 0 \geq \hat{u}(0, x) - u_0(x), \\ &\tilde{v}(0, x) - v_0(x) \geq 0 \geq \hat{v}(0, x) - v_0(x) \end{aligned} \qquad (x\in\Omega) \tag{3.16}$$

Since $(\bar{u}, 0)$ and $(\underline{u}, \bar{v})$ satisfy the equations

$$\begin{aligned} &\bar{u}_t - D_1\nabla^2\bar{u} = \bar{u}(a - b\bar{u}) \\ &\bar{v}_t - D_2\nabla^2\bar{v} = -c_1\underline{u}\bar{v} \\ &u\nabla^2\underline{u} = \underline{u}(a - b\underline{u} - c\bar{v}) \end{aligned}$$

and the boundary condition $B[\bar{u}] = B[\underline{u}] = B[\bar{v}] = 0] = B[\bar{v}] = 0$, all the differential and boundary inequalities in (3.14), (3.15) are fulfilled by $(\tilde{u}, \tilde{v}) = (\bar{u}, \bar{v})$, $(\hat{u}, \hat{v}) = (\underline{u}, 0)$. The initial requirements in (3.16) become

$$M_1 - u_0(x) \geq 0 \geq \delta_1\varphi - u_0(x),\ \delta_2 - v_0(x) \geq 0 \geq 0 - v_0(x)$$

By choosing $M_1 \geq u_0(x)$, $\delta_1 \leq \varepsilon$ in the definition of $\bar{u}$ and $\underline{u}$ the pair $(\bar{u}, \bar{v})$ and $(\underline{u}, 0)$ are upper and lower solutions provided that $v_0 \leq \delta_2$. With the restriction $v_0 \leq \delta_2$ the result in (2.4) holds for any $u_0 \geq \varepsilon\varphi$. To remove the restriction on v_0 we observe from $0 \leq v \leq V$ and $V(t, x) \to 0$ as $t \to \infty$ that for any $v_0 \geq 0$ there is a $t_1 > 0$ such that $v(t_1, x) \leq \delta_2$. By considering the problem (1.1) — (1.3) in the domain $[t_1, \infty) \times \Omega$ with the initial function $(\underline{u}(t_1, x), \bar{v}(t_1, x))$ and observing that $\underline{u}(t_1, x) \geq \delta_1\varphi \geq \varepsilon\varphi$ the solution $(\underline{u}, \bar{v})$ together with $(\bar{u}, 0)$ fulfill all the requirements in (3.14) (3.15) and (3.16) in $[t_1, \infty) \times \bar{\Omega}$. It follows again from the comparison theorem and Lemma 3.3 that the result in (2.4) holds. This completes the proof of the theorem.

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