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Efficient Algorithms for Approximating Particular Solutions of Elliptic Equations Using Chebyshev Polynomials

Andreas Karageorghis[1,*] and Irene Kyza[2]

[1] *Department of Mathematics and Statistics, University of Cyprus, P.O. Box 20537, 1678 Nicosia, Cyprus.*
[2] *Department of Mathematics, University of Crete, P.O. Box 2208, GR-714 09 Heraklion, Greece.*

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Abstract. In this paper, we propose efficient algorithms for approximating particular solutions of second and fourth order elliptic equations. The approximation of the particular solution by a truncated series of Chebyshev polynomials and the satisfaction of the differential equation lead to upper triangular block systems, each block being an upper triangular system. These systems can be solved efficiently by standard techniques. Several numerical examples are presented for each case.

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1 Introduction

Boundary methods such as the Boundary Integral Equation Method (BIEM) [2, 5] and the Method of Fundamental Solutions (MFS) [12,16] are numerical techniques applicable for the numerical solution certain elliptic boundary value problems. In these methods, the dimension of the problem is reduced by one as only the boundary of the domain of the problem under consideration needs to be discretized. The advantages of these techniques can be fully exploited if the governing differential equation is homogeneous. It is therefore often desirable to convert an elliptic boundary value problem governed by an inhomogeneous differential equation to one governed by a homogeneous differential equation. This can be achieved using the Method of Particular Solutions (MPS). To

*Corresponding author. *Email addresses:* `andreask@ucy.ac.cy` (A. Karageorghis), `kyza@math.uoc.gr` (I. Kyza)

describe the MPS, consider the boundary value problem

$$Lu=f \quad \text{in} \quad \Omega, \quad u=g \quad \text{on} \quad \partial\Omega, \tag{1.1}$$

where L is a second order linear elliptic operator and Ω is an open bounded domain in $\mathbb{R}^2$ with boundary $\partial\Omega$. If u_p is a particular solution of the governing equation, then it satisfies $Lu_p=f$ but does not necessarily satisfy the boundary condition. If we let $v=u-u_p$, then v satisfies the boundary value problem

$$Lv=0 \quad \text{in} \quad \Omega, \quad v=g-u_p \quad \text{on} \quad \partial\Omega. \tag{1.2}$$

Clearly, the governing equation is now homogeneous and thus problem (1.2) can be easily solved using a boundary-type method. In order to transform problem (1.1) into problem (1.2), we need to construct an approximation to the particular solution u_p.

In recent years, many methods have been proposed for the approximation of particular solutions. These methods may be classified as direct or indirect [10]. Direct methods approximate a solution of $Lu_p=f$ by a numerical method. For example, it is well-known that a particular solution of the Poisson equation $\Delta u_p=f$ in $\mathbb{R}^2$ is given by the Newtonian potential [1]

$$u_p(P)=\frac{1}{2\pi}\int_{\Omega}\log|P-Q|f(Q)dV(Q), \tag{1.3}$$

where $|P-Q|$ denotes the distance between the points P and Q. In general, the integral (1.3) cannot be evaluated analytically and so numerical integration is used. Since Ω can have an arbitrary shape, the numerical evaluation of the integral (1.3) requires a complicated domain discretization of Ω. To avoid the difficulties associated with such a discretization, Atkinson's method [1] may be used. In it, one assumes that f can be extended smoothly to $\tilde{\Omega}$, where $\Omega\subseteq\tilde{\Omega}$. Then $u_p(P)=\frac{1}{2\pi}\int_{\tilde{\Omega}}\log|P-Q|f(Q)dV(Q)$ is also a particular solution of $\Delta u_p=f$. The advantage of using this expression instead of (1.3) is that the domain $\tilde{\Omega}$ may be chosen so that the calculation of the integral is simplified [14]. The indirect approach for solving, for example, Poisson problems, is based on the Dual Reciprocity Method (DRM) [9,17,25]. In the DRM, the source term f is approximated by $\hat{f}=\sum_{i=1}^{n}a_i\hat{f}_i$, where $\{\hat{f}_i\}_{i=1}^{n}$ is an appropriate set of functions. An approximation to the particular solution u_p is obtained by taking $\hat{u}_p=\sum_{i=1}^{n}a_i\hat{u}_i$, where each $\hat{u}_i$ satisfies $\Delta\hat{u}_i=\hat{f}_i$. An appropriate set of functions is the set of Radial Basis Functions (RBFs) [7,9,15,17,18,22]. The most popular RBFs are thin plate and higher order radial splines, multiquadrics and Gaussians which are all globally supported [9,10,17–19]. The problem is that these globally supported basis functions lead to dense systems which can be highly ill-conditioned [9]. This difficulty can be overcome by using compactly supported RBFs (CS-RBFs) which have been extensively discussed in [9,17]. The most popular CS-RBFs are Wendland's CS-RBFs [9,17]. Polynomials and trigonometric functions have also been used as basis functions [22]. With these sets of basis functions a number of numerical methods can be used for determining approximation $\hat{f}$ [9,10,17,22]. The properties of orthogonal polynomials, such as Chebyshev and Legendre polynomials are well-documented [3,6,26]. It would therefore be reasonable to consider obtaining particular solutions by approximating f by truncated series of such polynomials on rectangular domains containing the actual domain of the problem. It is well-known that Chebyshev polynomials are valuable tools in numerical analysis and approximation theory [23]. In particular, they are widely used in the numerical solution of boundary value problems for partial differential equations with spectral methods [3,6,8,26]. The reason for this is the fact that a Chebyshev series expansion may be viewed as a cosine Fourier series, for if $f(x)=\sum_{n=0}^{\infty}a_n T_n(x)$ where $T_n(x)=\cos(n\cos^{-1}(x))$ denotes the Chebyshev polynomial of degree n on the interval $(-1,1)$, then $f(\cos\theta)=\sum_{n=0}^{\infty}a_n\cos(n\theta)$. This leads to rapid convergence properties and the possible use of Fast Fourier Transforms (FFTs) [6,26]. As far as the convergence of Chebyshev series is concerned, it can be shown [18,26] that if $p_N(x)=\sum_{n=0}^{N}a_n T_n(x)$ is the polynomial which interpolates a function $f\in C^s[-1,1]$, $s\geq 1$, at the Gauss-Lobatto points $x_n=\cos(n\pi/N)$, $n=0,1\cdots,N$, then p_N converges spectrally to f. This means that the more regular the function f, the more rapid the convergence of p_N to f. In particular, if f is infinitely differentiable, then the approximation error $||f-p_N||_\infty$ is smaller than any power of $1/N$ and the convergence is said to be exponential.

In this paper, we propose efficient algorithms for approximating particular solutions of second and fourth order elliptic equations using truncated series of Chebyshev polynomials. These algorithms could be viewed as Matrix Decomposition Algorithms (MDAs) for which an overview can be found in the survey article [4]. More recently, efficient Chebyshev spectral-Galerkin MDAs for second and fourth order elliptic boundary value problems were developed in [27]. Although the basis used in [27] consists of linear combinations of Chebyshev polynomials, the upper triangular matrix structures encountered in our approach is also present there. The MDA approach of [27] is extended to general ultraspherical polynomials in [11]. In this work we first consider the Poisson equation which is encountered in fluid mechanics, elasticity, heat conduction and is used to describe many other physical phenomena. We also consider the Helmholtz equation, which is encountered in acoustics, and the biharmonic equation, which is encountered in fluid mechanics and elasticity. The paper is organized as follows. In Section 2, we discuss the case of the Poisson equation. We first present the problem and the method which is based on some theoretical results. We then describe the algorithm in detail and apply it to several numerical examples. In Section 3, the algorithm is applied to the Helmholtz equation. In Section 4, we discuss the case of the biharmonic equation, describe the corresponding efficient algorithm and present numerical experiments. Finally, in Section 5, we provide some concluding remarks.

2 The Poisson equation

Our goal is to find an efficient way of obtaining an approximation of a particular solution of Poisson's equation on the rectangle $(a,b)\times(c,d)$, that is,

$$\Delta u(x,y)=f(x,y), \qquad (x,y)\in\Omega=(a,b)\times(c,d), \tag{2.1}$$

where the function $f(x,y)$ is known.

Let $T_m(x)$ and $T_n(y)$ denote the Chebyshev polynomials of degrees m and n, defined on the interval $(-1,1)$ in the x and y directions, respectively. We also let $T_m^x(x)$ and $T_n^y(y)$ denote the shifted Chebyshev polynomials of degrees m and n, defined on the intervals (a,b) and (c,d), respectively. Thus, $T_m^x(x)=T_m\left(\frac{2x-a-b}{b-a}\right)$, $T_n^y(y)=T_n\left(\frac{2y-c-d}{d-c}\right)$.

Further, suppose f is approximated by

$$f_{MN}(x,y)=\sum_{m=0}^{M}\sum_{n=0}^{N} f_{mn} T_m^x(x) T_n^y(y), \tag{2.2}$$

where the coefficients $f_{mn}, m=0,\cdots,M, n=0,\cdots,N$ are given by [18]

$$f_{mn}=\frac{4}{MN\bar{c}_m^x\bar{c}_n^y}\sum_{i=0}^{M}\sum_{j=0}^{N}\frac{f(x_i,y_j)}{\bar{c}_i^x\bar{c}_j^y}\cos\left(\frac{im\pi}{M}\right)\cos\left(\frac{jn\pi}{N}\right), \tag{2.3}$$

where $\bar{c}_0^x=\bar{c}_M^x=2$, $\bar{c}_i^x=1$, $i=1,\cdots,M-1$, $\bar{c}_0^y=\bar{c}_N^y=2$, $\bar{c}_i^y=1$, $i=1,\cdots,N-1$, and $x_i=(b-a)\xi_i/2+(b+a)/2$, $i=0,1,\cdots,M$, $y_j=(d-c)\eta_j/2+(d+c)/2$, $j=0,1,\cdots,N$, where $\xi_i=\cos(i\pi/M)$ and $\eta_j=\cos(j\pi/N)$ are sets of Gauss-Lobatto points on the interval $(-1,1)$. The quantities in expression (2.3) can be evaluated using FFTs.

Let $u(x,y)$ be approximated by $u_{MN}(x,y)$, where

$$u_{MN}(x,y)=\sum_{m=0}^{M}\sum_{n=0}^{N} u_{mn} T_m^x(x) T_n^y(y). \tag{2.4}$$

We want to find the coefficients $\{u_{mn}\}_{m,n=0}^{M,N}$ so that the Poisson equation $\triangle u_{MN}=f_{MN}$ is satisfied:

$$\sum_{m=0}^{M}\sum_{n=0}^{N} u_{mn} T_m^{x\,\prime\prime}(x) T_n^y(y)+\sum_{m=0}^{M}\sum_{n=0}^{N} u_{mn} T_m^x(x) T_n^{y\,\prime\prime}(y)=\sum_{m=0}^{M}\sum_{n=0}^{N} f_{mn} T_m^x(x) T_n^y(y) \tag{2.5}$$

for $(x,y)\in(a,b)\times(c,d)$. From [13], page 61, we have that

$$T_m^{x\,\prime}(x)=2m\frac{2}{(b-a)}\sum_{k=0}^{[\frac{m+1}{2}-1]} c_{m-2k-1} T_{m-2k-1}^x(x), \tag{2.6}$$

where $c_0=1/2$ and $c_i=1, i\in\mathbb{N}$. Clearly,

$$T_m^{x\,\prime\prime}(x)=2m\frac{4}{(b-a)^2}\sum_{k=0}^{[\frac{m+1}{2}-1]} c_{m-2k-1} T_{m-2k-1}^{x\,\prime}(x). \tag{2.7}$$

By substituting (2.6) into (2.7), it can be easily shown that [23]:

Proposition 2.1. *For m even,*

$$T_m^{x\,\prime\prime}(x)=\sum_{k=0}^{\frac{m}{2}-1}\alpha_{2k}^m T_{2k}^x(x), \tag{2.8}$$

where $\alpha_{2k}^m=4m(b-a)^{-2}c_{2k}(m^2-4k^2)$, $k=0,\cdots,m/2-1$. *For m odd,*

$$T_m^{x\,\prime\prime}(x)=\sum_{k=0}^{\frac{m-3}{2}}\alpha_{2k+1}^m T_{2k+1}^x(x), \tag{2.9}$$

where $\alpha_{2k+1}^m=4m(b-a)^{-2}c_{2k+1}(m^2-(2k+1)^2)$, $k=0,\cdots,(m-3)/2$.

Similarly, in the y-direction, for n even, we have

$$T_n^{y\,\prime\prime}(y)=\sum_{k=0}^{\frac{n}{2}-1}\beta_{2k}^n T_{2k}^y(y), \tag{2.10}$$

where $\beta_{2k}^n=4n(d-c)^{-2}c_{2k}(n^2-4k^2)$, $k=0,\cdots,n/2-1$, and, for n odd,

$$T_n^{y\,\prime\prime}(y)=\sum_{k=0}^{\frac{n-3}{2}}\beta_{2k+1}^n T_{2k+1}^y(y), \tag{2.11}$$

where $\beta_{2k+1}^n=4n(d-c)^{-2}c_{2k+1}(n^2-(2k+1)^2)$ $k=0,\cdots,(n-3)/2$. We can write

$$\sum_{m=0}^{M}\sum_{n=0}^{N}u_{mn}T_m^{x\,\prime\prime}(x)T_n^y(y)=\sum_{m=0}^{M-2}\sum_{n=0}^{N}v_{mn}T_m^x(x)T_n^y(y), \tag{2.12}$$

and similarly,

$$\sum_{m=0}^{M}\sum_{n=0}^{N}u_{mn}T_m^x(x)T_n^{y\,\prime\prime}(y)=\sum_{m=0}^{M}\sum_{n=0}^{N-2}w_{mn}T_m^x(x)T_n^y(y), \tag{2.13}$$

where the coefficients v_{mn} and w_{mn} can be expressed in terms of the coefficients u_{mn} from (2.8), (2.9) and (2.10), (2.11), respectively.

Thus (2.5) can be written as

$$\sum_{m=0}^{M-2}\sum_{n=0}^{N}v_{mn}T_m^x(x)T_n^y(y)+\sum_{m=0}^{M}\sum_{n=0}^{N-2}w_{mn}T_m^x(x)T_n^y(y)=\sum_{m=0}^{M}\sum_{n=0}^{N}f_{mn}T_m^x(x)T_n^y(y), \tag{2.14}$$

and, from (2.14), by equating the coefficients of $T_m^x(x)T_n^y(y)$, we obtain the system of equations,

$$v_{mn}+w_{mn}=f_{mn},\quad m=0,1,\cdots,M,\quad n=0,1,\cdots,N. \tag{2.15}$$

System (2.15) can be written as

$$(A\otimes I_N+I_M\otimes B)\mathbf{u}=\mathbf{f}, \tag{2.16}$$

where

$$\mathbf{u}^T=[u_{00},u_{01},u_{02},\cdots,u_{0N},u_{10},\cdots,u_{1N},\cdots,u_{M0},\cdots,u_{MN}],$$
$$\mathbf{f}^T=[f_{00},f_{01},f_{02},\cdots,f_{0N},f_{10},\cdots,f_{1N},\cdots,f_{M0},\cdots,f_{MN}],$$

and the symbol $\otimes$ denotes the matrix tensor product. When M is even,

$$A=\begin{pmatrix} 0 & 0 & \alpha_0^2 & 0 & \alpha_0^4 & 0 & \alpha_0^6 & \cdots & 0 & \alpha_0^M \\ 0 & 0 & 0 & \alpha_1^3 & 0 & \alpha_1^5 & 0 & \cdots & \alpha_1^{M-1} & 0 \\ 0 & 0 & 0 & 0 & \alpha_2^4 & 0 & \alpha_2^6 & \cdots & 0 & \alpha_2^M \\ 0 & 0 & 0 & 0 & 0 & \alpha_3^5 & 0 & \cdots & \alpha_3^{M-1} & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & \alpha_{M-3}^{M-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & \alpha_{M-2}^M \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}, \tag{2.17}$$

while, when M is odd,

$$A=\begin{pmatrix} 0 & 0 & \alpha_0^2 & 0 & \alpha_0^4 & 0 & \alpha_0^6 & \cdots & 0 & \alpha_0^{M-1} & 0 \\ 0 & 0 & 0 & \alpha_1^3 & 0 & \alpha_1^5 & 0 & \cdots & \alpha_1^{M-2} & 0 & \alpha_1^M \\ 0 & 0 & 0 & 0 & \alpha_2^4 & 0 & \alpha_2^6 & \cdots & 0 & \alpha_2^{M-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & \alpha_3^5 & 0 & \cdots & \alpha_3^{M-2} & 0 & \alpha_3^M \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & \alpha_{M-4}^{M-2} & 0 & \alpha_{M-4}^M \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & \alpha_{M-3}^{M-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \alpha_{M-2}^M \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \end{pmatrix}. \tag{2.18}$$

The form of the matrix B is the same as that of A, except that α is replaced by β and M is replaced by N. The $(M+1)(N+1)\times(M+1)(N+1)$ system (2.16) can be decomposed into the following four block systems:

$$\begin{aligned} B^*\mathbf{u}_0^*+\alpha_0^2\mathbf{u}_2^*+\alpha_0^4\mathbf{u}_4^*+ & \cdots + \alpha_0^{M_E}\mathbf{u}_{M_E}^* &&=\mathbf{f}_0^* \\ B^*\mathbf{u}_2^*+\alpha_2^4\mathbf{u}_4^*+ & \cdots + \alpha_2^{M_E}\mathbf{u}_{M_E}^* &&=\mathbf{f}_2^* \\ B^*\mathbf{u}_4^*+ & \cdots + \alpha_4^{M_E}\mathbf{u}_{M_E}^* &&=\mathbf{f}_4^* \\ \ddots \qquad & \qquad\quad \vdots && \quad\vdots \\ & B^*\mathbf{u}_{M_E-2}^*+\alpha_{M_E-2}^{M_E}\mathbf{u}_{M_E}^* &&=\mathbf{f}_{M_E-2}^* \\ & \qquad B^*\mathbf{u}_{M_E}^* &&=\mathbf{f}_{M_E}^* \end{aligned} \tag{2.19}$$

where $*=E$ and $*=O$ represent two block systems, and

$$\begin{array}{rcccl} B^*\mathbf{u}_1^*+\alpha_1^3\mathbf{u}_3^*+\alpha_1^5\mathbf{u}_5^*+ & \cdots & + & \alpha_1^{M_O}\mathbf{u}_{M_O}^* & =\mathbf{f}_1^* \\ B^*\mathbf{u}_3^*+\alpha_3^5\mathbf{u}_5^*+ & \cdots & + & \alpha_3^{M_O}\mathbf{u}_{M_O}^* & =\mathbf{f}_3^* \\ B^*\mathbf{u}_5^*+ & \cdots & + & \alpha_5^{M_O}\mathbf{u}_{M_O}^* & =\mathbf{f}_5^* \\ \ddots & & & \vdots & \vdots \\ & & & B^*\mathbf{u}_{M_O-2}^*+\alpha_{M_O-2}^{M_O}\mathbf{u}_{M_O}^* & =\mathbf{f}_{M_O-2}^* \\ & & & B^*\mathbf{u}_{M_O}^* & =\mathbf{f}_{M_O}^* \end{array} \tag{2.20}$$

where $*=E$ and $*=O$ represent the other two block systems, and where

$$\mathbf{u}_i^E=[u_{i0},u_{i2},\cdots,u_{iN_E}]^T,\quad \mathbf{u}_i^O=[u_{i1},u_{i3},\cdots,u_{iN_O}]^T,$$
$$\mathbf{f}_i^E=[f_{i0},f_{i2},\cdots,f_{iN_E}]^T,\quad \mathbf{f}_i^O=[f_{i1},f_{i3},\cdots,f_{iN_O}]^T,\quad i=0,1,\cdots M,$$

$$B^E=\begin{pmatrix} 0 & \beta_0^2 & \beta_0^4 & \beta_0^6 & \cdots & \beta_0^{N_E} \\ 0 & 0 & \beta_2^4 & \beta_2^6 & \cdots & \beta_2^{N_E} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & \beta_{N_E-2}^{N_E} \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix},\quad B^O=\begin{pmatrix} 0 & \beta_1^3 & \beta_1^5 & \cdots & \beta_1^{N_O} \\ 0 & 0 & \beta_3^5 & \cdots & \beta_3^{N_O} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \beta_{N_O-2}^{N_O} \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \tag{2.21}$$

and $M_E=2[M/2]$, $M_O=2[(M-1)/2]+1$, $N_E=2[N/2]$, $N_O=2[(N-1)/2]+1$. Note that $\mathbf{u}_i^E$ and $\mathbf{f}_i^E$ are $N_E/2+1$-vectors, $\mathbf{u}_i^O$ and $\mathbf{f}_i^O$ are $(N_O+1)/2$-vectors, the matrix B^E is $(N_E/2+1)\times(N_E/2+1)$ and the matrix B^O is $(N_O+1)/2\times(N_O+1)/2$.

Each of the systems (2.19)-(2.20) can be solved independently for $\mathbf{u}_i^E$, $i=0,2,\cdots,M_E$, $\mathbf{u}_i^O$, $i=0,2,\cdots,M_E$, $\mathbf{u}_i^E$, $i=1,3,\cdots,M_O$, and $\mathbf{u}_i^O$, $i=1,3,\cdots,M_O$, respectively. For instance, system (2.19) with $*=E$, can be solved by solving a sequence of $M_E/2+1$ upper triangular $(N_E/2+1)\times(N_E/2+1)$ subsystems. Each of these subsystems is underdetermined and we have addressed this issue by choosing $u_{i0}=0$, $i=0,2,\cdots,M_E$. Similarly, we choose $u_{i0}=0$, $i=1,3,\cdots,M_O$, $u_{i1}=0$, $i=0,2,\cdots,M_E$ and $u_{i1}=0$, $i=1,3,\cdots,M_O$. When choosing $u_{i0}=u_{i1}=0$, $i=0,1,\cdots,M$, the method is applicable for any values M and N. However, choosing the parameters u_{i0}, u_{i1}, $i=0,1,\cdots,M$, to be nonzero can, in certain cases, lead to inconsistencies. For instance, in the case $M=N$ is even, in subsystem (2.19) with $*=O$, we must take u_{M1} to be equal to zero in order to have consistency. The reason for this is the following. If we choose u_{M1} to be nonzero and solve the subsystem backwards, when we reach the first block line of (2.19), the last equation is not necessarily satisfied. On the other hand, in the case when $M=N$ is odd, there is no problem. It is therefore recommended to always choose $u_{i0}=u_{i1}=0$, $i=0,1,\cdots,M$. The cost of solving each upper triangular subsystem is clearly $\mathcal{O}(N^2)$ and thus the cost of solving each of systems (2.19) is $\mathcal{O}(MN^2)$. Each of the systems (2.19)-(2.20) is solved in a similar way and thus solving (2.16) has multiplicative complexity $\mathcal{O}(MN^2)$.

If we reorder the equations and unknowns, we can write (2.15) as

$$(I_N \otimes A + B \otimes I_M)\,\tilde{\mathbf{u}} = \tilde{\mathbf{f}}, \tag{2.22}$$

where

$$\tilde{\mathbf{u}}^T = [u_{00}, u_{10}, u_{20}, \cdots, u_{M0}, u_{01}, \cdots, u_{M1}, \cdots, u_{0N}, \cdots, u_{MN}],$$
$$\tilde{\mathbf{f}}^T = [f_{00}, f_{10}, f_{20}, \cdots, f_{M0}, f_{01}, \cdots, f_{M1}, \cdots, f_{0N}, \cdots, f_{MN}].$$

The $(M+1)(N+1)\times(M+1)(N+1)$ system (2.22) can be decomposed into the following four block systems:

$$\begin{array}{rcl} A^*\tilde{\mathbf{u}}_0^* + \beta_0^2\tilde{\mathbf{u}}_2^* + \beta_0^4\tilde{\mathbf{u}}_4^* + \cdots + \beta_0^{N_E}\tilde{\mathbf{u}}_{N_E}^* &=& \tilde{\mathbf{f}}_0^* \\ A^*\tilde{\mathbf{u}}_2^* + \beta_2^4\tilde{\mathbf{u}}_4^* + \cdots + \beta_2^{N_E}\tilde{\mathbf{u}}_{N_E}^* &=& \tilde{\mathbf{f}}_2^* \\ A^*\tilde{\mathbf{u}}_4^* + \cdots + \beta_4^{N_E}\tilde{\mathbf{u}}_{N_E}^* &=& \tilde{\mathbf{f}}_4^* \\ \ddots \qquad \vdots &\vdots& \\ A^*\tilde{\mathbf{u}}_{N_E-2}^* + \beta_{N_E-2}^{N_E}\tilde{\mathbf{u}}_{N_E}^* &=& \tilde{\mathbf{f}}_{N_E-2}^* \\ A^*\tilde{\mathbf{u}}_{N_E}^* &=& \tilde{\mathbf{f}}_{N_E}^* \end{array} \tag{2.23}$$

where $*=E$ and $*=O$ represent two block systems, and

$$\begin{array}{rcl} A^*\tilde{\mathbf{u}}_1^* + \beta_1^3\tilde{\mathbf{u}}_3^* + \beta_1^5\tilde{\mathbf{u}}_5^* + \cdots + \beta_1^{N_O}\tilde{\mathbf{u}}_{N_O}^* &=& \tilde{\mathbf{f}}_1^* \\ A^*\tilde{\mathbf{u}}_3^* + \beta_3^5\tilde{\mathbf{u}}_5^* + \cdots + \beta_3^{N_O}\tilde{\mathbf{u}}_{N_O}^* &=& \tilde{\mathbf{f}}_3^* \\ A^*\tilde{\mathbf{u}}_5^* + \cdots + \beta_5^{N_O}\tilde{\mathbf{u}}_{N_O}^* &=& \tilde{\mathbf{f}}_5^* \\ \ddots \qquad \vdots &\vdots& \\ A^*\tilde{\mathbf{u}}_{N_O-2}^* + \beta_{N_O-2}^{N_O}\tilde{\mathbf{u}}_{N_O}^* &=& \tilde{\mathbf{f}}_{N_O-2}^* \\ A^*\tilde{\mathbf{u}}_{N_O}^* &=& \tilde{\mathbf{f}}_{N_O}^* \end{array} \tag{2.24}$$

where $*=E$ and $*=O$ represent the other two block systems, and where

$$\tilde{\mathbf{u}}_j^E = [u_{0j}, u_{2j}, \cdots, u_{M_Ej}]^T, \quad \tilde{\mathbf{u}}_j^O = [u_{1j}, u_{3j}, \cdots, u_{M_Oj}]^T,$$
$$\tilde{\mathbf{f}}_j^E = [f_{0j}, f_{2j}, \cdots, f_{M_Ej}]^T, \quad \tilde{\mathbf{f}}_j^O = [f_{1j}, f_{3j}, \cdots, f_{M_Oj}]^T, \quad j=0,1,\cdots N,$$

$$A^E = \begin{pmatrix} 0 & \alpha_0^2 & \alpha_0^4 & \alpha_0^6 & \cdots & \alpha_0^{M_E} \\ 0 & 0 & \alpha_2^4 & \alpha_2^6 & \cdots & \alpha_2^{M_E} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & \alpha_{M_E-2}^{M_E} \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}, \quad A^O = \begin{pmatrix} 0 & \alpha_1^3 & \alpha_1^5 & \cdots & \alpha_1^{M_O} \\ 0 & 0 & \alpha_3^5 & \cdots & \alpha_3^{M_O} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \alpha_{M_O-2}^{M_O} \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}. \tag{2.25}$$

As in the previous ordering, each of the systems (2.23)-(2.24) can be solved independently for $\tilde{\mathbf{u}}_i^E$, $i=0,2,\cdots,M_E$, $\tilde{\mathbf{u}}_i^O$, $i=0,2,\cdots,M_E$, $\tilde{\mathbf{u}}_i^E$, $i=1,3,\cdots,M_O$, and $\tilde{\mathbf{u}}_i^O$, $i=1,3,\cdots,M_O$, respectively. Each of these systems is solved in a similar way to the way systems (2.19)-(2.20) were solved and thus solving system (2.22) has multiplicative complexity $\mathcal{O}(NM^2)$, having taken $\tilde{u}_{0j}=\tilde{u}_{1j}=0$, $j=0,1,\cdots,N$. As mentioned in the previous case, choosing the $\tilde{u}_{0j}$, $\tilde{u}_{1j}$, $j=0,1,\cdots,N$ to be different than zero can lead to inconsistencies and it is therefore recommended to always take these parameters to be equal to zero.

Both sets of coefficients $\mathbf{u}$ and $\tilde{\mathbf{u}}$ provide particular solutions of equation (2.1) via expression (2.4). Clearly, the vector $\mathbf{v}=(\mathbf{u}+\tilde{\mathbf{u}})/2$ also gives a particular solution of (2.1). For symmetry, we choose our particular solution to be $\mathbf{v}$. It is noteworthy that the determination of the coefficients u_{mn} from collocating equation (2.5) at $(M+1)(N+1)$ points has multiplicative complexity $\mathcal{O}(M^3N^3)$. A survey of Chebyshev collocation methods for Poisson problems may be found in [23].

We apply the proposed algorithm to a variety of functions $f(x,y)$ on $[a,b]\times[c,d]$. In order to demonstrate the accuracy of the method, we calculate the error

$$E=\max_{0\le i,j\le L}\left|\sum_{m=0}^{M}\sum_{n=0}^{N}v_{mn}T_m^{x\,\prime\prime}(x_i)T_n^y(y_j)+\sum_{m=0}^{M}\sum_{n=0}^{N}v_{mn}T_m^x(x_i)T_n^{y\,\prime\prime}(y_j)-f(x_i,y_j)\right|$$

on a uniform grid on $[a,b]\times[c,d]$ defined by $x_i=a+(b-a)(i/L)$, $i=0,1,\cdots,L$, $y_j=c+(d-c)(j/L)$, $j=0,1,\cdots,L$ where L was chosen to be equal to 100. For the evaluation of the Chebyshev polynomials and their derivatives at given points, we use the formulæ resulting from the definition of Chebyshev polynomials in terms of the cosine (see [6], Appendix A.2). In all examples we chose $M=N$.

We considered the following examples for which we present a graph of $\log E$ versus N (see Fig. 1):

Example 2.1 $f(x,y)=f_2(x,y)=\dfrac{1}{(4-x)(4-y)}$ on $[-1,1]\times[-1,1]$.

Example 2.2 $f(x,y)=f_4(x,y)=e^{xy}$ on $[-1,1]\times[-1,1]$.

Example 2.3 $f(x,y)=f_6(x,y)=\sin(3\pi x)\sin(3\pi y)$ on $[-0.2,0.5]\times[0.5,1]$.

We also investigate the conditioning of the matrices arising in systems (2.19)-(2.20) and (2.23)-(2.24). In particular, we calculate the condition numbers κ_∞ of the matrices A^E, A^O, B^E, B^O. These condition numbers were calculated using the NAG [24] routine `F07TGF`. In Fig. 2, we present the condition numbers κ_∞ of the matrices A^E, A^O defined on the interval $(-1,1)$.

From the numerical results, we observe that, for all the examples considered, the Chebyshev approximation converges exponentially to the exact f as N grows. It should be noted that, for large values of M and N, there is loss of accuracy due to ill-conditioning of the coefficient matrices of the systems involved (see Fig. 2). Further examples confirming these observations may be found in [20,21].

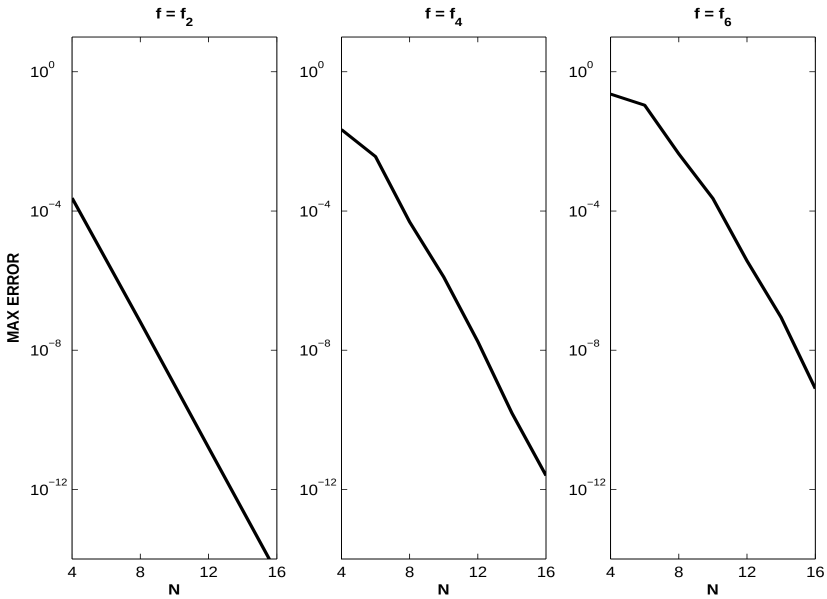

Figure 1: Maximum error versus N for Examples 2.1-2.3.

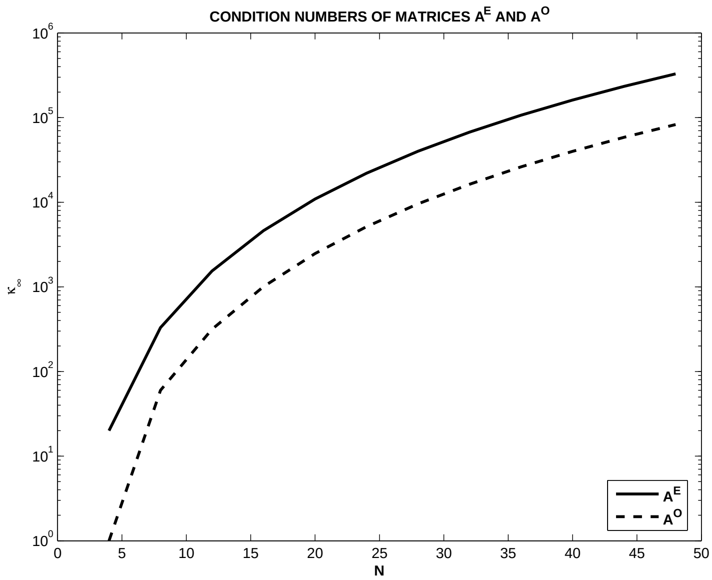

Figure 2: Condition number κ_∞ versus N for matrices A^E and A^O on $(-1,1)$.

3 The Helmholtz equation

In this case, our goal is to find an efficient way of approximating a particular solution of the Helmholtz equation on the rectangle $(a,b)\times(c,d)$, that is,

$$\Delta u(x,y)+ku(x,y)=f(x,y), \qquad (x,y)\in\Omega=(a,b)\times(c,d), \tag{3.1}$$

where $k\neq 0$ is a constant and the function $f(x,y)$ is known. In the case $k>0$ equation (3.1) is known as the Helmholtz equation whereas if $k<0$, it is known as the *modified* Helmholtz equation. Adopting the notation used in Section 2 and following the same steps as in the case of the Poisson equation, we want the coefficients $\{u_{mn}\}_{m,n=0}^{M,N}$ in (2.4) to satisfy

$$\begin{aligned}&\sum_{m=0}^{M}\sum_{n=0}^{N}u_{mn}T_m^{x\prime\prime}(x)T_n^y(y)+\sum_{m=0}^{M}\sum_{n=0}^{N}u_{mn}T_m^x(x)T_n^{y\prime\prime}(y)+k\sum_{m=0}^{M}\sum_{n=0}^{N}u_{mn}T_m^x(x)T_n^y(y)\\&=\sum_{m=0}^{M}\sum_{n=0}^{N}f_{mn}T_m^x(x)T_n^y(y)\end{aligned}\tag{3.2}$$

for $(x,y)\in(a,b)\times(c,d)$. Using (2.12) and (2.13), (3.2) becomes

$$\begin{aligned}&\sum_{m=0}^{M-2}\sum_{n=0}^{N}v_{mn}T_m^x(x)T_n^y(y)+\sum_{m=0}^{M}\sum_{n=0}^{N-2}w_{mn}T_m^x(x)T_n^y(y)+k\sum_{m=0}^{M}\sum_{n=0}^{N}u_{mn}T_m^x(x)T_n^y(y)\\&=\sum_{m=0}^{M}\sum_{n=0}^{N}f_{mn}T_m^x(x)T_n^y(y),\end{aligned}\tag{3.3}$$

and by equating the coefficients of the products of Chebyshev polynomials $T_m^x(x)T_n^y(y)$, $m=0,1,\cdots,M$, $n=0,1,\cdots,N$, we obtain the system of equations $v_{mn}+w_{mn}+ku_{mn}=f_{mn}$, $m=0,1,\cdots,M$, $n=0,1,\cdots,N$. This system can be written as

$$(A\otimes I_N+I_M\otimes B+kI_M\otimes I_N)\,\mathbf{u}=\mathbf{f},\tag{3.4}$$

where $A,B,\mathbf{u}$ and $\mathbf{f}$ are defined as in Section 2. It can be decomposed into the four block systems (2.19)-(2.20) where now the matrices B^E and B^O are replaced by $\hat{B}^E$ and $\hat{B}^O$, respectively, which are of the same forms as those of B^E and B^O as defined in (2.21) except that the diagonal values 0 are replaced by k. As in the Poisson case, each of the four systems can be solved independently for $\mathbf{u}_i^E$, $i=0,2,\cdots,M_E$, $\mathbf{u}_i^O$, $i=0,2,\cdots,M_E$, $\mathbf{u}_i^E$, $i=1,3,\cdots,M_O$ and $\mathbf{u}_i^O$, $i=1,3,\cdots,M_O$, respectively. However, each of these subsystems is no longer underdetermined and therefore we do not need to preassign any coefficients. As before, the cost of solving each upper triangular subsystem is clearly $\mathcal{O}(N^2)$ and thus the cost of solving the four block systems is $\mathcal{O}(MN^2)$.

If we reorder the equations and unknowns we can write the system as

$$(I_N\otimes A+B\otimes I_M+kI_N\otimes I_M)\,\tilde{\mathbf{u}}=\tilde{\mathbf{f}},\tag{3.5}$$

where $\tilde{\mathbf{u}}$ and $\tilde{\mathbf{f}}$ are defined as in the Poisson case. This system can be decomposed into the four block systems (2.23)-(2.24) where now the matrices A^E and A^O are replaced by $\hat{A}^E$ and $\hat{A}^O$, respectively, which are of the same forms as those of A^E and A^O as defined in (2.25) except that the diagonal values 0 are replaced by k. Each system can be solved independently for $\tilde{\mathbf{u}}_i^E$, $i=0,2,\cdots,M_E$, $\tilde{\mathbf{u}}_i^O$, $i=0,2,\cdots,M_E$, $\tilde{\mathbf{u}}_i^E$, $i=1,3,\cdots,M_O$ and $\tilde{\mathbf{u}}_i^O$, $i=$

$1,3,\cdots,M_O$, respectively. Each of these subsystems is nonsingular and therefore we do not need to preassign any coefficients.

Both sets of coefficients $\mathbf{u}$ and $\tilde{\mathbf{u}}$ provide particular solutions of (3.1) via expression (2.4). As in the Poisson case, for reasons of symmetry, we choose our particular solution to be $\mathbf{v}=(\mathbf{u}+\tilde{\mathbf{u}})/2$.

With the same notation as in the Poisson case, but with the error now defined as

$$E=\max_{0\le i,j\le L}\left|\sum_{m=0}^{M}\sum_{n=0}^{N}v_{mn}T_m^{x\prime\prime}(x_i)T_n^y(y_j)+\sum_{m=0}^{M}\sum_{n=0}^{N}v_{mn}T_m^x(x_i)T_n^{y\prime\prime}(y_j)\right.$$

$$\left.+k\sum_{m=0}^{M}\sum_{n=0}^{N}v_{mn}T_m^x(x_i)T_n^y(y_j)-f(x_i,y_j)\right|,$$

we consider the following examples (with $k=1$ and $M=N$) for which we present a graph of $\log E$ versus N (see Fig. 3):

Example 3.1 $f(x,y)=f_7(x,y)=e^{x+y}$ on $[-1,1]\times[-1,1]$.

Example 3.2 $f(x,y)=f_2(x,y)=\dfrac{1}{(4-x)(4-y)}$ on $[-1,1]\times[-1,1]$.

From the numerical results, we observe that the Chebyshev approximation converges exponentially to the exact f as N grows. It should be noted, however, that for moderate values of M and N, there is significant loss of accuracy due to ill-conditioning of the coefficient matrices of the systems involved. The ill-conditioning in the Helmholtz case is considerably more serious than in the Poisson case as can be observed from Fig. 4, where we present the condition number κ_∞ of the matrices A^E and A^O defined on the interval $(-1,1)$, corresponding to the Helmholtz case. Further examples may be found in [21].

4 The biharmonic equation

Our goal is to find a particular solution of the biharmonic equation on the rectangle $(a,b)\times(c,d)$, that is

$$\Delta^2u(x,y)=f(x,y),\qquad (x,y)\in\Omega=(a,b)\times(c,d), \tag{4.1}$$

where the function $f(x,y)$ is known. Using the notation of Section 2, we approximate $u(x,y)$ by

$$u_{MN}(x,y)=\sum_{m=0}^{M}\sum_{n=0}^{N}u_{mn}T_m^x(x)T_n^y(y), \tag{4.2}$$

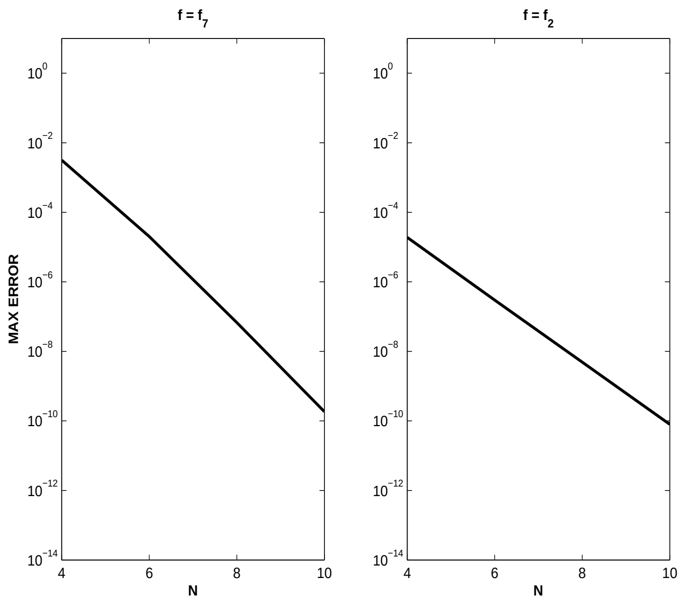

Figure 3: Maximum error versus N for Examples 3.1 and 3.2 in Helmholtz case.

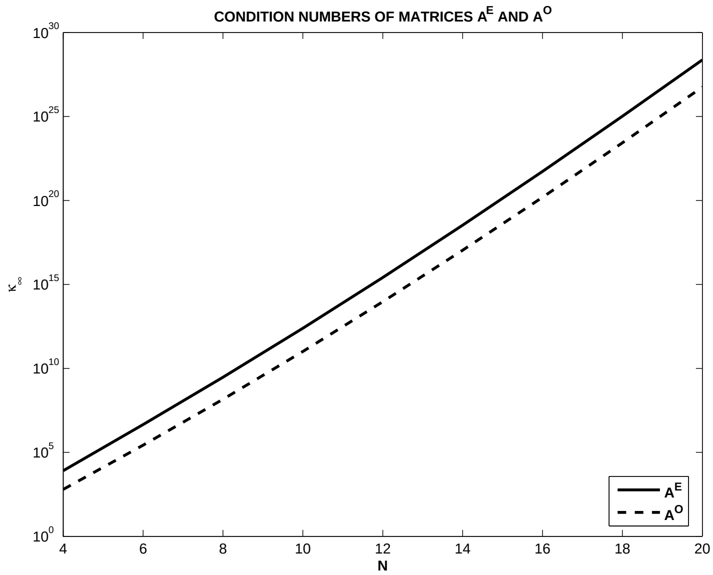

Figure 4: Condition number κ_∞ versus N for matrices A^E and A^O on (-1,1) in Helmholtz case.

where the coefficients $\{u_{mn}\}_{m,n=0}^{M,N}$ are determined so that the following equation is satisfied:

$$\sum_{m=0}^{M}\sum_{n=0}^{N} u_{mn} T_m^{x\,''''}(x) T_n^y(y) + 2\sum_{m=0}^{M}\sum_{n=0}^{N} u_{mn} T_m^{x\,''}(x) T_n^{y\,''}(y) + \sum_{m=0}^{M}\sum_{n=0}^{N} u_{mn} T_m^x(x) T_n^{y\,''''}(y)$$
$$= \sum_{m=0}^{M}\sum_{n=0}^{N} f_{mn} T_m^x(x) T_n^y(y), \qquad (x,y)\in(a,b)\times(c,d). \tag{4.3}$$

Using (2.8), in the case m is even with, say, $m=2t$,

$$T_m^{x\,''''}(x)=\sum_{k=0}^{\frac{m}{2}-1}\alpha_{2k}^m T_{2k}^{x\,''}(x)=\sum_{k=1}^{\frac{m}{2}-1}\alpha_{2k}^m\left(\sum_{\ell=0}^{k-1}\alpha_{2\ell}^{2k}T_{2\ell}^x(x)\right)=\sum_{k=0}^{\frac{m}{2}-2}\gamma_{2k}^m T_{2k}^x(x), \tag{4.4}$$

and rearranging the series, we obtain

$$\begin{aligned}\gamma_{2k}^m=&\sum_{\ell=k+1}^{\frac{m}{2}-1}\alpha_{2\ell}^m\alpha_{2k}^{2\ell}=32c_{2k}m\sum_{\ell=k+1}^{\frac{m}{2}-1}\ell\left(\frac{m^2}{4}-\ell^2\right)\left(\ell^2-k^2\right)\\ =&64c_{2k}t\sum_{\ell=k+1}^{t-1}\ell\left(t^2-\ell^2\right)\left(\ell^2-k^2\right),\quad k=0,1,\ldots,t-2.\end{aligned} \tag{4.5}$$

Similarly, in the case m is odd with, say, $m=2t+1$, using (2.9) we have,

$$\begin{aligned}T_m^{x\,''''}(x)=&\sum_{k=0}^{\frac{m-3}{2}}\alpha_{2k+1}^m T_{2k+1}^{x\,''}(x)\\ =&\sum_{k=1}^{\frac{m-3}{2}}\alpha_{2k+1}^m\left(\sum_{\ell=0}^{k-1}\alpha_{2\ell+1}^{2k+1}T_{2\ell+1}^x(x)\right)=\sum_{k=0}^{\frac{m-3}{2}}\gamma_{2k+1}^m T_{2k+1}^x(x).\end{aligned} \tag{4.6}$$

Rearrangement of the series yields

$$\begin{aligned}\gamma_{2k+1}^m=&\sum_{\ell=k+1}^{\frac{m-3}{2}}\alpha_{2\ell+1}^m\alpha_{2k+1}^{2\ell+1}=m\sum_{\ell=k+1}^{\frac{m-3}{2}}(2\ell+1)\left(m^2-(2\ell+1)^2\right)\left((2\ell+1)^2-(2k+1)^2\right)\\ =&(2t+1)\sum_{\ell=k+1}^{t-1}(2\ell+1)\left((2t+1)^2-(2\ell+1)^2\right)\left((2\ell+1)^2-(2k+1)^2\right),\end{aligned} \tag{4.7}$$

where $k=0,1,\ldots,t-2$.

From (4.4)-(4.5) and (4.6)-(4.7), we have the following result:

Proposition 4.1. *For even $m=2t$,*

$$T_m^{x\,''''}(x)=\sum_{k=0}^{\frac{m}{2}-2}\gamma_{2k}^m T_{2k}^x(x), \tag{4.8}$$

where $\gamma_{2k}^m=256(b-a)^{-4}c_{2k}t\left((t-k)^2-1\right)\left(t^2-k^2\right)\left((t+k)^2-1\right)/3$ for $0\le k\le t-2$. For odd $m=2t+1$, we have

$$T_m^{x\,''''}(x)=\sum_{k=0}^{\frac{m-5}{2}}\gamma_{2k+1}^m T_{2k+1}^x(x), \tag{4.9}$$

where $\gamma_{2k+1}^m=128(b-a)^{-4}(2t+1)\left((t-k)^2-1\right)\left(t^2-k^2\right)(t+k+1)(t+k+2)/3$ for $0\le k\le t-2$.

Similarly, it can also be shown that for even $n=2t$,

$$T_n^{y''''}(y)=\sum_{k=0}^{\frac{n}{2}-2}\delta_{2k}^n T_{2k}^y(y), \tag{4.10}$$

where $\delta_{2k}^n=256(d-c)^{-4}c_{2k}t\left((t-k)^2-1\right)\left(t^2-k^2\right)\left((t+k)^2-1\right)/3$ for $0\le k\le t-2$, while for odd $n=2t+1$ we have

$$T_n^{y''''}(y)=\sum_{k=0}^{\frac{n-5}{2}}\delta_{2k+1}^n T_{2k+1}^y(y), \tag{4.11}$$

where $\delta_{2k+1}^n=128(d-c)^{-4}(2t+1)\left((t-k)^2-1\right)\left(t^2-k^2\right)(t+k+1)(t+k+2)/3$ for $0\le k\le t-2$. We can write

$$\sum_{m=0}^{M}\sum_{n=0}^{N}u_{mn}T_m^{x''}(x)T_n^{y''}(y)=\sum_{m=0}^{M-2}\sum_{n=0}^{N-2}z_{mn}T_m^x(x)T_n^y(y), \tag{4.12}$$

where the coefficients z_{mn} can be expressed in terms of the coefficients u_{mn} from (2.8), (2.9), (2.10) and (2.11). Similarly, we can write

$$\sum_{m=0}^{M}\sum_{n=0}^{N}u_{mn}T_m^{x''''}(x)T_n^y(y)=\sum_{m=0}^{M-4}\sum_{n=0}^{N}v_{mn}T_m^x(x)T_n^y(y), \tag{4.13}$$

and

$$\sum_{m=0}^{M}\sum_{n=0}^{N}u_{mn}T_m^x(x)T_n^{y''''}(y)=\sum_{m=0}^{M}\sum_{n=0}^{N-4}w_{mn}T_m^x(x)T_n^y(y), \tag{4.14}$$

where the coefficients v_{mn} and w_{mn} can be expressed in terms of the u_{mn} from (4.8), (4.9) and (4.10), (4.11), respectively. Thus (4.3) can be written as

$$\begin{aligned}&\sum_{m=0}^{M-4}\sum_{n=0}^{N}v_{mn}T_m^x(x)T_n^y(y)+2\sum_{m=0}^{M-2}\sum_{n=0}^{N-2}z_{mn}T_m^x(x)T_n^y(y)+\sum_{m=0}^{M}\sum_{n=0}^{N-4}w_{mn}T_m^x(x)T_n^y(y)\\&=\sum_{m=0}^{M}\sum_{n=0}^{N}f_{mn}T_m^x(x)T_n^y(y),\end{aligned} \tag{4.15}$$

and by equating the coefficients of $T_m^x(x)T_n^y(y)$, we obtain the system of equations

$$v_{mn}+2z_{mn}+w_{mn}=f_{mn},\quad m=0,1,\cdots,M,\quad n=0,1,\cdots,N. \tag{4.16}$$

System (4.16) can be written as

$$(C\otimes I_N+2A\otimes B+I_M\otimes D)\,\mathbf{u}=\mathbf{f}, \tag{4.17}$$

where

$$\begin{aligned}\mathbf{u}^T&=[u_{00},u_{01},u_{02},\cdots,u_{0N},u_{10},\cdots,u_{1N},\cdots,u_{M0},\cdots,u_{MN}],\\ \mathbf{f}^T&=[f_{00},f_{01},f_{02},\cdots,f_{0N},f_{10},\cdots,f_{1N},\cdots,f_{M0},\cdots,f_{MN}],\end{aligned}$$

and the matrices A and B are given by (2.17)-(2.18). When M is even,

$$C=\begin{pmatrix} 0 & 0 & 0 & 0 & \gamma_0^4 & 0 & \gamma_0^6 & 0 & \cdots & 0 & \gamma_0^M \\ 0 & 0 & 0 & 0 & 0 & \gamma_1^5 & 0 & \gamma_1^7 & \cdots & \gamma_1^{M-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \gamma_2^6 & 0 & \cdots & 0 & \gamma_2^M \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \gamma_3^7 & \cdots & \gamma_3^{M-1} & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & \gamma_{M-5}^{M-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & \gamma_{M-4}^M \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}, \tag{4.18}$$

while when M is odd,

$$C=\begin{pmatrix} 0 & 0 & 0 & 0 & \gamma_0^4 & 0 & \gamma_0^6 & 0 & \cdots & 0 & \gamma_0^{M-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & \gamma_1^5 & 0 & \gamma_1^7 & \cdots & \gamma_1^{M-2} & 0 & \gamma_1^M \\ 0 & 0 & 0 & 0 & 0 & 0 & \gamma_2^6 & 0 & \cdots & 0 & \gamma_2^{M-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \gamma_3^7 & \cdots & \gamma_3^{M-2} & 0 & \gamma_3^M \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & \gamma_{M-4}^{M-2} & 0 & \gamma_{M-4}^M \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & \gamma_{M-3}^{M-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \gamma_{M-2}^M \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \end{pmatrix}. \tag{4.19}$$

The form of the matrix D is the same as that of C, except that γ is replaced by δ and M is replaced by N.

The $(M+1)(N+1)\times(M+1)(N+1)$ system (4.17) can be decomposed into the following four systems:

$$\begin{aligned} D^*\mathbf{u}_0^*+2\alpha_0^2B^*\mathbf{u}_2^*+(\gamma_0^4I+2\alpha_0^4B^*)\mathbf{u}_4^*+\cdots+(\gamma_0^{M_E}I+2\alpha_0^{M_E}B^*)\mathbf{u}_{M_E}^*&=\mathbf{f}_0^* \\ D^*\mathbf{u}_2^*+2\alpha_2^4B^*\mathbf{u}_4^*+\cdots+(\gamma_2^{M_E}I+2\alpha_2^{M_E}B^*)\mathbf{u}_{M_E}^*&=\mathbf{f}_2^* \\ D^*\mathbf{u}_4^*+\cdots+(\gamma_4^{M_E}I+2\alpha_4^{M_E}B^*)\mathbf{u}_{M_E}^*&=\mathbf{f}_4^* \\ &\vdots \\ D^*\mathbf{u}_{M_E-2}^*+2\alpha_{M_E-2}^{M_E}B^*\mathbf{u}_{M_E}^*&=\mathbf{f}_{M_E-2}^* \\ D^*\mathbf{u}_{M_E}^*&=\mathbf{f}_{M_E}^* \end{aligned} \tag{4.20}$$

where $*=E$ and $*=O$ represent two block systems, and

$$\begin{aligned} D^*\mathbf{u}_1^*+2\alpha_1^3B^*\mathbf{u}_3^*+(\gamma_1^5I+2\alpha_1^5B^*)\mathbf{u}_5^*+\cdots+(\gamma_1^{M_O}I+2\alpha_1^{M_O}B^*)\mathbf{u}_{M_O}^*&=\mathbf{f}_1^*\\ D^*\mathbf{u}_3^*+2\alpha_3^5B^*\mathbf{u}_5^*+\cdots+(\gamma_3^{M_O}I+2\alpha_3^{M_O}B^*)\mathbf{u}_{M_O}^*&=\mathbf{f}_3^*\\ D^*\mathbf{u}_5^*+\cdots+(\gamma_5^{M_O}I+2\alpha_5^{M_O}B^*)\mathbf{u}_{M_O}^*&=\mathbf{f}_5^*\\ &\vdots\\ D^*\mathbf{u}_{M_O-2}^*+2\alpha_{M_O-2}^{M_O}B^*\mathbf{u}_{M_O}^*&=\mathbf{f}_{M_O-2}^*\\ D^*\mathbf{u}_{M_O}^*&=\mathbf{f}_{M_O}^* \end{aligned} \tag{4.21}$$

where $*=E$ and $*=O$ represent the other two block systems, and where

$$\mathbf{u}_i^E=[u_{i0},u_{i2},\cdots,u_{iN_E}]^T,\quad \mathbf{u}_i^O=[u_{i1},u_{i3},\cdots,u_{iN_O}]^T,$$
$$\mathbf{f}_i^E=[f_{i0},f_{i2},\cdots,f_{iN_E}]^T,\quad \mathbf{f}_i^O=[f_{i1},f_{i3},\cdots,f_{iN_O}]^T,\quad i=0,1,\cdots M,$$

the matrices B^E and B^O are given by (2.21), and

$$D^E=\begin{pmatrix} 0&0&\delta_0^4&\delta_0^6&\cdots&\delta_0^{N_E}\\ 0&0&0&\delta_2^6&\cdots&\delta_2^{N_E}\\ \vdots&\vdots&\vdots&\vdots&\ddots&\vdots\\ 0&0&0&\cdots&0&\delta_{N_E-4}^{N_E}\\ 0&0&0&\cdots&0&0\\ 0&0&0&\cdots&0&0 \end{pmatrix},\quad D^O=\begin{pmatrix} 0&0&\delta_1^5&\delta_1^7&\cdots&\delta_1^{N_O}\\ 0&0&0&\delta_3^7&\cdots&\delta_3^{N_O}\\ \vdots&&\vdots&\vdots&\ddots&\vdots\\ 0&0&0&0&\cdots&\delta_{N_O-4}^{N_O}\\ 0&0&0&0&\cdots&0\\ 0&0&0&0&\cdots&0 \end{pmatrix}, \tag{4.22}$$

and the dimensions M_E, M_O, N_E and N_O are given in Section 2. Note that $\mathbf{u}_i^E$ and $\mathbf{f}_i^E$ are $N_E/2+1$-vectors, $\mathbf{u}_i^O$ and $\mathbf{f}_i^O$ are $(N_O+1)/2$-vectors, the matrix B^E is $(N_E/2+1)\times(N_E/2+1)$ and the matrix B^O is $(N_O+1)/2\times(N_O+1)/2$.

Each of the systems (4.20)-(4.21) can be solved independently for $\mathbf{u}_i^E$, $i=0,2,\cdots,M_E$, $\mathbf{u}_i^O$, $i=0,2,\cdots,M_E$, $\mathbf{u}_i^E$, $i=1,3,\cdots,M_O$ and $\mathbf{u}_i^O$, $i=1,3,\cdots,M_O$, respectively. For instance, (4.20) with $*=E$, can be solved by solving a sequence of $M_E/2+1$ upper triangular $(N_E/2+1)\times(N_E/2+1)$ subsystems. Each of these subsystems is underdetermined and we therefore choose $u_{i0}=u_{i2}=0$, $i=0,2,\cdots,M_E$. Similarly, we choose $u_{i1}=u_{i3}=0$, $i=0,2,\cdots,M_E$, $u_{i0}=u_{i2}=0$, $i=1,3,\cdots,M_O$ and $u_{i1}=u_{i3}=0$, $i=1,3,\cdots,M_O$. Choosing these parameters to be nonzero can lead to inconsistencies as in the Poisson case, and it is therefore recommended to always choose $u_{i0}=u_{i1}=u_{i2}=u_{i3}=0$, $i=0,1,\cdots,M$. The cost of solving each upper triangular subsystem is clearly $\mathcal{O}(N^2)$ and thus the cost of solving each of systems (4.20) is $\mathcal{O}(MN^2)$. Each of the systems (4.20)-(4.21) is solved in a similar way and thus solving (4.17) has a cost of $\mathcal{O}(MN^2)$.

If we reorder the equations and unknowns, we can write (4.16) as

$$(I_N\otimes C+2B\otimes A+D\otimes I_M)\,\tilde{\mathbf{u}}=\tilde{\mathbf{f}}, \tag{4.23}$$

where

$$\tilde{\mathbf{u}}^T = [u_{00}, u_{10}, u_{20}, \cdots, u_{M0}, u_{01}, \cdots, u_{M1}, \cdots, u_{0N}, \cdots, u_{MN}],$$
$$\tilde{\mathbf{f}}^T = [f_{00}, f_{10}, f_{20}, \cdots, f_{M0}, f_{01}, \cdots, f_{M1}, \cdots, f_{0N}, \cdots, f_{MN}].$$

The $(M+1)(N+1)\times(M+1)(N+1)$ system (4.23) can be decomposed into four independent subsystems as in the previous ordering. Each of these systems is solved independently as described earlier for systems (4.20)-(4.21) and thus solving system (4.23) has a cost of $\mathcal{O}(NM^2)$. The full details of this decomposition may be found in [20].

Both sets of coefficients $\mathbf{u}$ and $\tilde{\mathbf{u}}$ provide particular solutions of equation (4.1) via expression (4.2). Clearly, the vector $\mathbf{v}=(\mathbf{u}+\tilde{\mathbf{u}})/2$ also gives a particular solution of (4.1). For symmetry, we choose our particular solution to be $\mathbf{v}$. We applied the proposed algorithm to a variety of functions $f(x,y)$ on $[a,b]\times[c,d]$.

In order to demonstrate the accuracy of the method, we calculate the error

$$E = \max_{0\le i,j\le L} \left| \sum_{m=0}^{M}\sum_{n=0}^{N} v_{mn} \left(T_m^{x\,''''}(x_i) T_n^y(y_j) + 2T_m^{x\,''}(x_i) T_n^{y\,''}(y_j) + T_m^x(x_i) T_n^{y\,''''}(y_j) \right) - f(x_i,y_j) \right|$$

on a uniform grid on $[a,b]\times[c,d]$ defined in Section 2.

We consider the following examples (with $M=N$) for which we present a graph of $\log E$ versus N (see Fig. 5):

Example 4.1 $f(x,y)=f_8(x,y)=\sin(x)\sin(y)$ on $[-1,1]\times[-1,1]$.

Example 4.2 $f(x,y)=f_4(x,y)=e^{xy}$ on $[-1,1]\times[-1,1]$.

Example 4.3 $f(x,y)=f_9(x,y)=\sinh(x+y)$ on $[-0.5,1.5]\times[0.5,2]$.

As in the case of second order problems, we observe that for all the examples considered, the Chebyshev approximation converges exponentially to the exact f as N increases. As is those, however, we observed that, for large values of M and N, there is loss of accuracy due to ill-conditioning of the coefficient matrices of the systems involved (see Fig. 6). Additional numerical examples may be found in [20].

5 Concluding remarks

In this work, we use Chebyshev polynomial expansions to estimate approximations to particular solutions of second and fourth order elliptic partial differential equations in two dimensions. By using elementary properties of Chebyshev polynomials, it is shown that the methods used, lead, in both cases, to the solution of four block systems. Each of these systems can be solved independently, by solving a sequence of upper triangular subsystems. The cost of solving each of the four block systems has multiplicative

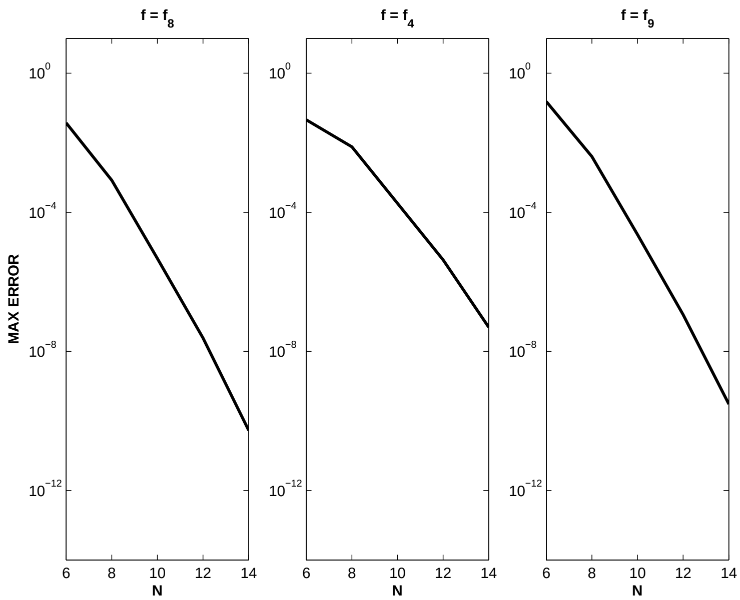

Figure 5: Maximum error versus N for Examples 4.1-4.3 in biharmonic case.

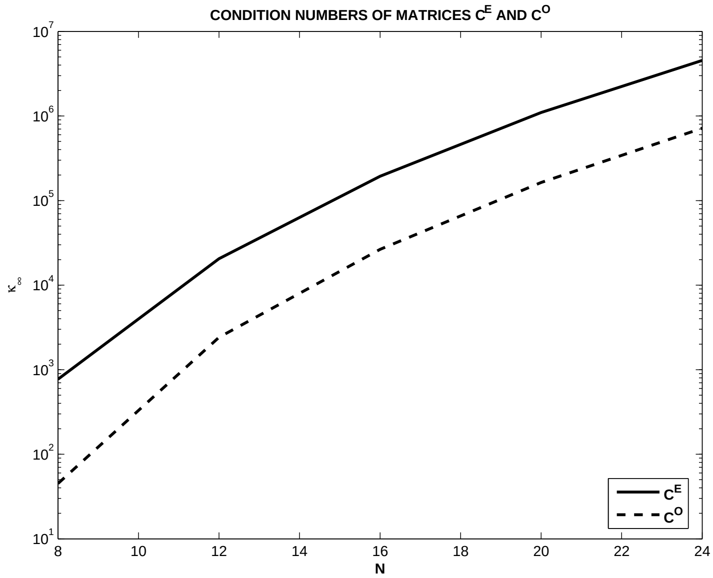

Figure 6: Condition number κ_∞ versus N for matrices C^E and C^O on (-1,1) in biharmonic case.

complexity $\mathcal{O}(\mathcal{N}^3)$ and thus the cost of the algorithm for calculating the Chebyshev approximation is also $\mathcal{O}(\mathcal{N}^3)$ where $\mathcal{N}$ is the highest degree of the Chebyshev polynomials used in each direction. The methods are applied to several numerical examples and it is shown that the errors for both types of equations converge exponentially to zero with $\mathcal{N}$. In the case of the Poisson equation, however, it was observed that for large values of $\mathcal{N}$ the accuracy of the method suffered due to the ill-conditioning of the matrices involved. In the case of the biharmonic equation, this problem was slightly exacerbated and the conditioning of the matrices became poorer for smaller values of $\mathcal{N}$. This is an interesting phenomenon and requires further investigation. The algorithms developed in this work maybe used in the MPS for the solution of inhomogeneous boundary value problems with boundary methods, such as the MFS. The application of this technique to three-dimensional problems is currently under investigation.

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