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Hedging Game Contingent Claims with Constrained Portfolios

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Abstract. Game option is an American-type option with added feature that the writer can exercise the option at any time before maturity. In this paper, we consider the problem of hedging Game Contingent Claims (GCC) in two cases. For the case that portfolio is unconstrained, we provide a single arbitrage-free price P_0. Whereas for the constrained case, the price is replaced by an interval $[h_{low}, h_{up}]$ of arbitrage-free prices. And for the portfolio with some closed constraints, we give the expressions of the upper-hedging price and lower-hedging price. Finally, for a special type of game option, we provide explicit expressions of the price and optimal portfolio for the writer and holder.

AMS subject classifications: 60G40, 91A60

Key words: Game option, contingent claims, hedging, optimal stopping, free boundary.

1 Introduction

In [2], Kifer introduced Game option, and he also gave the expressions of the value process and optimal stopping times for the holder and writer. Then many authors continue the research, in this direction, see, e.g., [4–6, 8–11] and so on. However, most of the research focuses on the deduction of the expressions and properties of the price. In this paper, we will mainly consider the problem of hedging Game Contingent Claims (GCC).

In section 2 we briefly give some background information, including market model and some definitions. In section 3 we discuss the portfolio without constrains and point out that the upper– and lower–hedging prices both equal to a given arbitrage-free price P_0. Section 4 we begin to study portfolio constraints, where we mainly investigate the hedging problem for the GCC under general portfolio constraints, and

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give an arbitrage-free interval $[h_{low}, h_{up}]$. Section 5 is the main part of this paper. In this section we consider the constraints on portfolio and obtain the expressions of upper- and lower-hedging prices based on the introduction of an auxiliary family $\{\mathcal{M}_\nu\}_{\nu\in\mathcal{D}}$. In addition, we also point out that there exists a hedging portfolio. In section 6, we give some examples.

2 Market model

Consider the Black-Scholes market $\mathcal{M}$. That is, there is only one risky asset S and a riskless bond B. They satisfy

$$dB_t = B_t r_t dt, \qquad B_0 = 1, \tag{2.1}$$
$$dS_t = S_t[r_t dt + \sigma_t dW_t], \quad S_0 = x \in (0,\infty), \tag{2.2}$$

respectively, with W a standard Brownian motion on the complete probability space $(\Omega, \mathcal{F}, \mathbf{P})$, and we will denote its natural filtration

$$\mathcal{F}_t^W = \sigma(W_s : 0 \le s \le t), \quad 0 \le t \le T < \infty, \quad \text{by} \quad \mathbf{F} = \{\mathcal{F}_t\}_{0\le t\le T}.$$

Here the process r_t is interest rate, σ_t is volatility. We suppose that they are all $\mathbf{F}$-progressively measurable and bounded uniformly in $[0,T]\times\Omega$. Furthermore, $\sigma_t(\omega)$ will be assumed to be invertible, with $\sigma_t^{-1}(w)$ bounded in $[0,T]\times\Omega$. Let

$$\beta_t = 1/B_t = \exp\Big\{-\int_0^t r_s\, ds\Big\}.$$

Then

$$d\beta_t S_t = \beta_t S_t \sigma_t dW_t. \tag{2.3}$$

That is, under $\mathbf{P}$ the discounted stock price $\beta_t S_t \triangleq \widetilde{S}_t$ is a martingale, so we will call $\mathbf{P}$ an equivalent martingale measure.

Definition 2.1. (1) *An* $\mathbf{F}$*-progressively measurable process* $\pi : [0,T]\times\Omega \to \mathbf{R}$ *with*

$$\int_0^t \pi_t^2 dt < \infty \text{ a.s.},$$

is called a portfolio process. (2) *An* $\mathbf{F}$*-adapted process* $C : [0,T]\times\Omega \to \mathbf{R}$ *with increasing, right continuous paths and*

$$C_0 = 0, \quad C_T < \infty, \quad \text{a.s.},$$

is called cumulative consumption process. We will call (π, C) *portfolio consumption process.*

Definition 2.2. *For any given portfolio consumption process pair* (π, C), $x \in \mathbf{R}$, *the solution*

$$V \equiv V^{x,\pi,C},$$

of the linear stochastic equation

$$dV_t = \pi_t \frac{dS_t}{S_t} + (V_t - \pi_t)\frac{dB_t}{B_t} - dC_t = r_t V_t dt + \pi_t \sigma_t dW_t - dC_t, \quad V_0 = x, \tag{2.4}$$

is called the wealth process corresponding to initial capital x, portfolio rule π and cumulative consumption rule C.

Here π_t represents the amount of the agent's wealth that is invested in the stock at time t, and this amount may be negative, which means that short-selling of the stock is permitted. Corresponding, $V_t - \pi_t$ represents the amount that is put into the bank account, and this amount can also take negative values, which means that borrowing from the bank at the interest rate r_t is permitted. If we define

$$p_t \triangleq \begin{cases} \pi_t / V_t, & V_t \neq 0, \\ 0, & V_t = 0, \end{cases}, \quad \theta_t^i \triangleq \begin{cases} \pi_t / S_t = p_t V_t / S_t, & i = 1, \\ \dfrac{V_t - \pi_t}{B_t} = \dfrac{V_t(1-p_t)}{B_t}, & i = 0. \end{cases} \tag{2.5}$$

Then p_t represents the proportions of wealth that is invested in the stock at time t, while $\theta_t^1(\theta_t^0)$ represents the number-of-shares held in the stock (account) at time t. Thus we obtain

$$V_t = \theta_t^0 B_t + \theta_t^1 S_t, \quad 0 \le t \le T. \tag{2.6}$$

The solution of (2.4) is given by

$$\beta_t V_t + \int_0^t \beta_s \, dC_s = x + \int_0^t \beta_s \, \pi_s \, \sigma_s \, dW_s = x + \int_0^t \beta_s \, p_s V_s \, \sigma_s \, dW_s. \tag{2.7}$$

Definition 2.3. *We say that a portfolio consumption process pair (π, C) is admissible in $\mathcal{M}$ for the initial wealth x, if there exists a constant $c \geq 0$ such that*

$$V_t \equiv V_t^{x,\pi,C} \geq -c, \quad 0 \le t \le T. \tag{2.8}$$

We shall denote by $\mathcal{A}_0(x)$ the class of all such pairs.

We call $\widetilde{V}_t \triangleq \beta_t V_t$ discount wealth process, from Def. 2.3 we can define its admissibility similarly and it is obvious that they are equal to each other. If (π, C) is a admissible strategy, then from (2.7) we know that $\widetilde{V}_t$ is a $\mathbf{P}$–supermartingale.

For $0 \le s \le t \le T$, we shall denote by $\mathcal{S}_{s,t}$ the class of $\mathbf{F}$-stopping times that values in $[s, t]$. Specially we let

$$\mathcal{S} \equiv \mathcal{S}_{0,T}.$$

For any given $\tau \in \mathcal{S}$, we denote by $\mathcal{A}_0(x, \tau)$ the class of portfolio consumption process pair (π, C) for which the stopped process $V_{t\wedge\tau}^{x,\pi,C}$ satisfies the requirement (2.8). Clearly we have

$$\mathcal{A}_0(x) = \mathcal{A}_0(x, T) \subseteq \mathcal{A}_0(x, \tau).$$

3 Game contingent claims in an unconstrained market

Suppose that $X=\{X_t : t \leq T\}$ and $Y=\{Y_t : t \leq T\}$ be two continuous stochastic process defined on $(\Omega, \mathcal{F}, \mathbf{P})$ such that for all $0\leq t\leq T$, $X_t\leq Y_t$ a.s.. The game option is a contract between a holder and writer at time $t=0$. It is a general American-type option with the added property that the writer has the right to terminate the contract at any time before expiry time T. If the holder exercises first, then (s)he may obtain the value of X at the exercise time and if the writer exercise first, then (s)he is obliged to pay to the holder the value of Y at the time of exercise. If neither has exercised at time T and $T<\infty$, then the writer pays the holder the value X_T. If both decide to claim at the same time then the lesser of the two claims is paid. In short, if the holder will exercise with strategy τ and the writer with strategy γ, we can conclude that at any moment during the life of the contract, the holder can expect to receive

$$Z(\tau,\gamma) \triangleq X_\tau 1_{(\tau\leq\gamma)} + Y_\gamma 1_{(\gamma<\tau)}.$$

We will call $Z(\cdot,\cdot)$ game contingent claims. In addition, suppose that for some $\epsilon>0$, we have

$$\mathbf{E}\Big(\sup_{0\leq t\leq T}(\beta_t Y_t)^{1+\epsilon}\Big) < \infty. \tag{3.1}$$

In order to obtain this commitment, the holder has to pay an amount $x\geq 0$ to the writer, and this amount is the price of the contingent claim. Then what should this amount be? In other words, what is the fair price? Here the so-called "fair" is nothing but satisfying the requirement of the writer and holder. So we should consider the problem from the point of view of both sides. For the writer, after receiving the amount $x\geq 0$, (s)he hope that (s)he can find a stopping time $\gamma\in\mathcal{S}$ and a portfolio consumption pair $(\pi, C)\in\mathcal{A}_0(x,\gamma)$, such that, whatever who terminates the contract first, he can fulfil his obligation without risk. That is, for any $0\leq t\leq T$, we have

$$V^{x,\pi,C}_{t\wedge\gamma} \geq Z(t,\gamma) \text{ a.s.}. \tag{3.2}$$

We will call this triple (γ,π,C) the writer's upper-hedging portfolio. The smallest value of initial capital $x\geq 0$ that allows the writer to do this is called upper hedging price for the GCC, that is

$$h_{up} \triangleq \inf\Big\{x\geq 0 :\ \exists\,\gamma\in\mathcal{S},\ (\pi,C)\in\mathcal{A}_0(x,\gamma),\ \text{s.t. (3.2) holds}\Big\}. \tag{3.3}$$

We will call the triple (γ^*,π^*,C^*) the writer's optimal upper hedging strategy if it attains h_{up}.

Now, from the point of view of the holder, after paying an amount $x\geq 0$ to the writer, (s)he wants to find a stopping time $\tau\in\mathcal{S}$ and a portfolio consumption (π',C') $\in\mathcal{A}_0(-x,\tau)$, such that, whatever who terminates the contract first, (s)he can fill up the deficit for purchasing this contingent claim. That is, for any $0\leq t\leq T$, we have

$$V^{-x,\pi',C'}_{\tau\wedge t} + Z(\tau,t) \geq 0 \text{ a.s.}. \tag{3.4}$$

The largest amount $x \geq 0$ that enables the holder to do this is called lower hedging price for the GCC, that is

$$h_{low} \triangleq \sup \Big\{ x \geq 0 : \ \exists \, \tau \in \mathcal{S}, \ (\pi', C') \in \mathcal{A}_0(-x, \tau), \ \text{s.t. (3.4) holds} \Big\}. \tag{3.5}$$

We will call the triple (τ, π', C') the holder's hedging portfolio and (τ^*, π_*, C_*) the holder's optimal hedging portfolio if it attains h_{low}.

From definition, we can understand that h_{up} is the smallest amount of initial capital required by the writer in order to find a portfolio consumption process pair that would guarantee his wealth at any time exceeds his liability. While for h_{low}, it is the largest initial capital the holder needs to borrow in order to find a portfolio consumption process pair that would guarantee his debt is covered at the time of execution of the contract.

Theorem 3.1. *Suppose that $T<\infty$ is finite, then the value process $P=\{P_t : t \in [0,T]\}$ of the game option is given by*

$$\begin{aligned} P_t &= \operatorname*{ess\,sup}_{\tau \in \mathcal{S}_{t,T}} \operatorname*{ess\,inf}_{\gamma \in \mathcal{S}_{t,T}} \frac{1}{\beta_t} \mathbf{E}^0 \left[\beta_{\tau \wedge \gamma} Z(\tau, \gamma) | \mathcal{F}_t \right] \\ &= \operatorname*{ess\,inf}_{\tau \in \mathcal{S}_{t,T}} \operatorname*{ess\,sup}_{\gamma \in \mathcal{S}_{t,T}} \frac{1}{\beta_t} \mathbf{E}^0 \left[\beta_{\tau \wedge \gamma} Z(\tau, \gamma) | \mathcal{F}_t \right]. \end{aligned}$$

The optimal stopping strategies for the holder and writer respectively are

$$\tau^* = \inf\{u \geq t : P_u = X_u\} \wedge T, \quad \gamma^* = \inf\{u \geq t : P_u = Y_u\} \wedge T. \tag{3.6}$$

For the upper-hedging price and lower-hedging price in this unconstrained market, we have following theorem and the proof will be omitted.

Theorem 3.2. $h_{low} = P_0 = h_{up}$.

4 Portfolio with constraints

In the previous section, we primarily consider the problem of hedging GCC and do not constrain the portfolio, that is, we can borrow from the bank and sell stock without limit. In fact, it is not realistic, so we need some conditions to restrict the portfolio. Karatzas consider the problem of hedging American contingent claim (ACC), following him and Suppose that K_+ and K_- are two Borel subsets of $\mathbf{R}$, each of which contains the origin. We restrict attention to portfolio consumption rules (π, C) that satisfy

$$\begin{cases} p_t \in K_+, & \text{if} \quad V_t^{x,\pi,C} > 0 \\ p_t \in K_-, & \text{if} \quad V_t^{x,\pi,C} < 0 \end{cases},$$

then admissible portfolio consumption process pairs becomes

$$\mathcal{A}(x) \triangleq \left\{ (\pi, C) \in \mathcal{A}_0(x) \left\{ \begin{array}{ll} p_t \in K_+, & \text{on } \{V_t^{x,\pi,C} > 0\} \\ p_t \in K_-, & \text{on } \{V_t^{x,\pi,C} < 0\} \end{array} \right\}, \ \forall\, 0 \le t \le T \right\}. \tag{4.1}$$

We shall also consider the subclasses

$$\mathcal{A}_+(x) \triangleq \Big\{ (\pi, C) \in \mathcal{A}(x) : p_t \in K_+,\ V_t^{x,\pi,C} \ge 0 \ \text{ a.s., } \forall\, 0 \le t \le T,\ x \ge 0 \Big\}, \tag{4.2}$$

$$\mathcal{A}_-(x) \triangleq \Big\{ (\pi, C) \in \mathcal{A}(x) : p_t \in K_-,\ V_t^{x,\pi,C} \le 0 \ \text{ a.s., } \forall\, 0 \le t \le T,\ x \le 0 \Big\}. \tag{4.3}$$

The definition of $\mathcal{A}(x,\tau)$ and $\mathcal{A}_\pm(x,\tau)$ is similar as before. We shall denote by $\mathcal{M}(K)$ the market $\mathcal{M}$ with constraints, then the upper-hedging price and lower-hedging price become

$$\begin{aligned} h_{up}(K) &\triangleq \inf\{x \ge 0 : \exists\, \gamma \in \mathcal{S} \text{ and } (\pi, C) \in \mathcal{A}_+(x,\gamma), \text{ s.t. (3.2) holds}\} \\ &\triangleq \inf A, \end{aligned} \tag{4.4}$$

$$\begin{aligned} h_{low}(K) &\triangleq \sup\{x \ge 0 : \exists\, \tau \in \mathcal{S} \text{ and } (\pi', C') \in \mathcal{A}_-(-x,\tau), \text{ s.t. (3.4) holds}\} \\ &\triangleq \sup B. \end{aligned} \tag{4.5}$$

Here we have

$$0 \le X_0 \le h_{low}(K) \le P_0 \le h_{up}(K). \tag{4.6}$$

However, $h_{low}(K){=}h_{up}(K)$ does not hold for this case that the portfolio has constraint. In fact, we have $h_{low}(K){<}h_{up}(K)$. In order to depict these two prices, let us first give a definition.

Definition 4.1. *Suppose that $p{>}0$ is the price of GCC at time $t{=}0$, we say that the market $\mathcal{M}(K)$ admits an arbitrage opportunity, if there exists either*

(i) a stopping time $\gamma{\in}\mathcal{S}$ and a portfolio consumption pair $(\pi, C){\in}\mathcal{A}_+(x,\gamma)$, such that

$$V_{t\wedge\gamma}^{x,\pi,C} \ge Z(t,\gamma), \tag{4.7}$$

holds for some $0{<}x{<}p$; or

(ii) a stopping time $\tau{\in}\mathcal{S}$ and a portfolio consumption pair $(\pi', C'){\in}\mathcal{A}_-(-x,\tau)$, such that

$$V_{\tau\wedge t}^{-x,\pi',C'} + Z(\tau, t) \ge 0, \tag{4.8}$$

holds for some $x{>}p$.

Where in the first case, after receiving the amount $p\,(p > x)$, the writer can use some(e.g. x) to hedge it without risk, then the difference of this two amount can be the arbitrage obtained by the writer. For the second case, if the holder buy the contingent claim with the amount x, then (s)he can find a portfolio consumption to hedge it without risk. While in fact, the amount of capital used to buy the GCC is only $p\,(p < x)$, this difference can bring to the holder an arbitrage opportunity.

Theorem 4.1. *For every $p>0$, if $p\notin[h_{low}(K),h_{up}(K)]$, then there exists an arbitrage opportunity in this market. When $p>h_{up}(K)$, this opportunity belongs to the writer, and when $p<h_{low}(K)$, it belongs to the holder. However, when $p\in[h_{low}(K),h_{up}(K)]$, there is no arbitrage opportunity, so we will call this interval the arbitrage-free interval.*

5 Portfolio with closed constraints

In the previous section we constrain the portfolio, now let us strengthen the condition. Suppose that both K_+ and K_- are closed interval of $\mathbf{R}$, that $K_+\cap K_-$ contains the origin. In addition, for any $p_+\in K_+$, $p_-\in K_-$, we have

$$\lambda p_+ + (1-\lambda)p_- \in \begin{cases} K_+ & \text{if } \lambda\geq 1, \\ K_- & \text{if } \lambda<0. \end{cases} \tag{5.1}$$

The support function of $-K_+$ is given by

$$\delta(y)\triangleq \sup_{p\in K_+}(-py): \mathbf{R}\to[0,\infty], \tag{5.2}$$

and its effective domain is

$$\begin{aligned}\widetilde{K}&\triangleq\Big\{y\in\mathbf{R}: \exists\beta\in\mathbf{R}, \text{ s.t. } -py\leq\beta, \forall p\in K_+\Big\}\\ &=\Big\{y\in\mathbf{R}:\delta(y)<\infty\Big\}.\end{aligned} \tag{5.3}$$

(5.1) guarantees that the sets $-K_+$ and K_- have the same effective domain, that is

$$\widetilde{K}=\{y\in\mathbf{R}:\exists\beta\in\mathbf{R}, \text{ s.t. } py\leq\beta, \forall p\in K_-\}, \tag{5.4}$$

and

$$\sup_{p\in K_-}(py)=\begin{cases} -\delta(y), & y\in\widetilde{K}, \\ \infty, & y\notin\widetilde{K}. \end{cases} \tag{5.5}$$

Finally we shall assume that $\delta(y)$ is continuous on $\widetilde{K}$.

Let $\mathcal{H}$ be the space of $\mathbf{F}$-progressively measurable processes ν: $[0,T]\times\Omega\to\widetilde{K}$ which satisfies

$$\mathbf{E}\int_0^T\left[\nu^2(t)+\delta(\nu(t))\right]dt<\infty. \tag{5.6}$$

And the auxiliary family $\{\mathcal{M}_\nu\}$ of random environment parametrized by processes $\nu(\cdot)$ contains the market $\mathcal{M}$, then for the choice $\nu\equiv0$, we have $\mathcal{M}_0=\mathcal{M}$. Thus, for each member $\mathcal{M}_\nu$ of this family, we can use the former method to solve the pricing problem for the GCC. In the new market model, define

$$\beta^\nu(t)\triangleq\exp\left\{-\int_0^t[r(s)+\delta(v(s))]ds\right\},\quad \theta^\nu(t)\triangleq\sigma^{-1}(t)\nu(t), \tag{5.7}$$

$$\begin{aligned}Z^\nu(t)&\triangleq\exp\left\{-\int_0^t\theta^\nu(s)dW(s)-\frac{1}{2}\int_0^t(\theta^\nu(s))^2ds\right\},\\ W^\nu(t)&\triangleq W(t)+\int_0^t\sigma^{-1}(s)\nu(s)ds.\end{aligned} \tag{5.8}$$

Let $\mathcal{D}$ denote the bounded processes in $\mathcal{H}$, and define

$$\mathbf{P}^{\nu}(A)=\mathbf{E}[Z^{\nu}(T)1_A], \quad A\in\mathcal{F}_T.$$

Then the process $Z^{\nu}(\cdot)$ and $W^{\nu}(\cdot)$ are martingale and Brownian motion respectively under this measure $\mathbf{P}^{\nu}$, and in this new market model $\mathcal{M}_{\nu}$, the price of the GCC is given by

$$P_{\nu}(0)=\sup_{\tau\in\mathcal{S}}\inf_{\gamma\in\mathcal{S}}\mathbf{E}^{\nu}[\beta^{\nu}(\tau\wedge\gamma)Z(\tau,\gamma)], \tag{5.9}$$

as both the upper-hedging and lower-hedging price of GCC in this new market with unconstrained portfolios. Combining (5.7) and (5.8), from (2.7) we obtain

$$\begin{aligned}&\beta^{\nu}(t)V(t)+\int_0^t\beta^{\nu}(s)dC_s+\int_0^t\beta^{\nu}(s)V(s)[\delta(\nu(s))+\nu(s)p(s)]ds\\ &=x+\int_0^t\beta^{\nu}(s)V(s)p(s)\sigma(s)dW^{\nu}(s).\end{aligned} \tag{5.10}$$

For every $\nu\in\mathcal{D}$, let

$$V_{\nu}(t)\triangleq\frac{1}{\beta^{\nu}(t)}\operatorname*{ess\,sup}_{\tau\in\mathcal{S}_{t,T}}\operatorname*{ess\,inf}_{\gamma\in\mathcal{S}_{t,T}}\mathbf{E}^{\nu}[\,\beta^{\nu}(\tau\wedge\gamma)Z(\tau,\gamma)|\mathcal{F}_t].$$

This is the value process of the GCC in $\mathcal{M}_{\nu}$. It is obvious that

$$V_{\nu}(0)=P_{\nu}(0), \quad V_{\nu}(T)=X(T).$$

In addition, we introduce the processes

$$\overline{V}(t)=\operatorname*{ess\,sup}_{\nu\in\mathcal{D}}V_{\nu}(t), \quad P\triangleq\sup_{\nu\in\mathcal{D}}V_{\nu}(0)=\sup_{\nu\in\mathcal{D}}\sup_{\tau\in\mathcal{S}}\inf_{\gamma\in\mathcal{S}}\mathbf{E}^{\nu}[\,\beta^{\nu}(\tau\wedge\gamma)Z(\tau,\gamma)],$$

and define stopping time

$$\gamma_t=\inf\{u\in[t,T):\overline{V}(u)=Y(u)\}\wedge T. \tag{5.11}$$

Lemma 5.1. *For every fixed $\nu\in\mathcal{D}$ and stopping times $\tau\le\rho\le\gamma_0$, we have*

$$\beta^{\nu}(\tau)\overline{V}(\tau)\ge\mathbf{E}^{\nu}[\beta^{\nu}(\rho)\overline{V}(\rho)|\mathcal{F}_{\tau}] \quad \text{a.s.}. \tag{5.12}$$

$\overline{V}(t\wedge\gamma_0)$ can be considered in its RCLL modification. So under the measure $\mathbf{P}^{\nu}$, the process

$$Q_{\nu}(t)\triangleq\beta^{\nu}(t\wedge\gamma_0)\overline{V}(t\wedge\gamma_0), \quad 0\le t\le T,$$

is a supermartingale.

Theorem 5.1. *The upper-hedging price $h_{up}(K)$ is given by*

$$h_{up}(K) = P \triangleq \sup_{\nu\in\mathcal{D}} P_\nu(0) = \sup_{\nu\in\mathcal{D}} \sup_{\tau\in\mathcal{S}} \inf_{\gamma\in\mathcal{S}} \mathbf{E}^\nu[\beta^\nu(\tau\wedge\gamma)Z(\tau,\gamma)]. \tag{5.13}$$

Furthermore, if $P<\infty$, there exists a pair $(\hat{\pi},\hat{C}) \in \mathcal{A}_+(P,\gamma_0)$, such that

$$V^{P,\hat{\pi},\hat{C}}(t\wedge\gamma_0) = \overline{V}(t\wedge\gamma_0) \geq Z(t,\gamma_0) \quad \text{a.s.,} \tag{5.14}$$

with γ_0 the definition of (5.11).

Proof. First, let us prove the inequality $P\leq h_{up}(K)$. It is obvious if $h_{up}(K)=\infty$. If not, that is, $h_{up}(K)<\infty$, then the set A defined by (4.4) is nonempty. Let $0\leq x\in A$ be an arbitrary element of this set, $\hat{\gamma}\in\mathcal{S}$ be a stopping time, and $(\hat{\pi},\hat{C})\in\mathcal{A}_+(x,\hat{\gamma})$ be any portfolio consumption process that satisfies (3.2). Under $\mathbf{P}^\nu$, the process of (5.10) is a nonnegative local martingale and then a supermartingale. Thus, for any $\tau\in\mathcal{S}$ and $\nu\in\mathcal{D}$, from (5.2) and (3.2) we have

$$\begin{aligned} x \geq\ & \mathbf{E}^\nu\Big[\beta^\nu(\tau\wedge\hat{\gamma})V^{x,\hat{\pi},\hat{C}}(\tau\wedge\hat{\gamma}) + \int_0^{\tau\wedge\hat{\gamma}} \beta^\nu(s)dC_s + \int_0^{\tau\wedge\hat{\gamma}} \beta^\nu(s)\Big(\delta(\nu_s)+\nu(s)\hat{p}(s)\Big)ds\Big] \\ \geq\ & \mathbf{E}^\nu\Big[\beta^\nu(\tau\wedge\hat{\gamma})V^{x,\hat{\pi},\hat{C}}(\tau\wedge\hat{\gamma})\Big] \geq \mathbf{E}^\nu\Big[\beta^\nu(\tau\wedge\hat{\gamma})Z(\tau\wedge\hat{\gamma})\Big]. \end{aligned}$$

The second inequality follows from the fact that $\hat{p}(t)\in K_+$ and the definition of $\delta(x)$, where $\hat{p}(t)$ is given by (2.5), that is

$$\hat{p}\,(t) = \begin{cases} \hat{\pi}(t)/V^{x,\hat{\pi},\hat{C}}(t), & \text{if } V^{x,\hat{\pi},\hat{C}}(t) > 0, \\ 0, & \text{if } V^{x,\hat{\pi},\hat{C}}(t) = 0. \end{cases}$$

Thus,

$$x \geq \sup_{\nu\in\mathcal{D}} \sup_{\tau\in\mathcal{S}} \inf_{\gamma\in\mathcal{S}} = \mathbf{E}^\nu[\beta^\nu(\tau\wedge\gamma)Z(\tau,\gamma)] = P,$$

and $h_{up}(K)\geq P$ follows from the arbitrariness of x.

Now we will prove $h_{up}(K)\leq P$. If $P=\infty$, then the inequality is obvious. Suppose that $P<\infty$. If there exists a stopping time $\hat{\gamma}\in\mathcal{S}$ and a pair $(\hat{\pi},\hat{C})\in\mathcal{A}_+(P,\hat{\gamma})$ which satisfies (5.14), we can obtain $P\in A$ and $h_{up}(K)\leq P$. Lemma 5.1 implies that for any $\nu\in\mathcal{D}$, $Q_\nu(t)$ is a $\mathbf{P}^\nu$-supermartingale, then Doob-Meyer decomposition gives

$$Q_\nu(t) = P + M_\nu(t) - A_\nu(t), \tag{5.15}$$

where

$$M_\nu(t) = \int_0^t \psi_\nu(s)dW^\nu(s), \quad 0\leq t\leq T,$$

is a $\mathbf{P}^\nu$-martingale,

$$\psi_\nu : [0,T]\times\Omega \to \mathbf{R},$$

is an **F**-progressively measurable process with $\int_0^T \psi_\nu^2(t)dt<\infty$ a.s., $A_\nu(\cdot)$ is an **F**-adapted increasing process with right-continuous paths and $A_\nu(0)=0$, $\mathbf{E}^\nu A_\nu(T)<\infty$. Because (5.15) holds for any $\nu\in\mathcal{D}$, so we may take $\psi_\nu(\cdot)\equiv 0$, a.e. on stochastic interval $[\![\gamma_0, T]\!]$ and $A_\nu(T)=A_\nu(\gamma_0)$ a.s.. For any $\mu\in\mathcal{D}$, from (5.15) we obtain

$$\begin{aligned} Q_\mu(t) = P &+ \int_0^t \frac{\beta^\mu(s)}{\beta^\nu(s)}\psi_\nu(s)dW^\mu(s) - \int_0^t \frac{\beta^\mu(s)}{\beta^\nu(s)}dA_\nu(s) \\ &- \int_0^t \frac{\beta^\mu(s)}{\beta^\nu(s)}\Big[\psi_\nu(s)\sigma^{-1}(s)\big(\mu(s)-\nu(s)\big) + \beta^\nu(s)\overline{V}(s)\big(\delta(\mu_s)-\delta(\nu_s)\big)\Big]ds, \end{aligned} \tag{5.16}$$

and

$$Q_\mu(t) = P + \int_0^t \psi_\mu(s)dW^\mu(s) - A_\mu(t). \tag{5.17}$$

Comparing (5.16) and (5.17) we obtain that the processes

$$\frac{\psi_\nu(t)}{\beta^\nu(t)} = \frac{\psi_\mu(t)}{\beta^\mu(t)} \triangleq h(t), \tag{5.18}$$

$$\begin{aligned} &\int_0^t \frac{dA_\mu(s)}{\beta^\mu(s)} - \int_0^t \Big(\overline{V}(s)\delta(\mu_s) + h(s)\sigma^{-1}(s)\mu(s)\Big)ds \\ =&\int_0^t \frac{dA_\nu(s)}{\beta^\nu(s)} - \int_0^t \Big(\overline{V}(s)\delta(\nu_s) + h(s)\sigma^{-1}(s)\nu(s)\Big)ds \triangleq \hat{C}(t), \end{aligned} \tag{5.19}$$

do not depend on $\nu\in\mathcal{D}$. Specially, if we take $\nu(\cdot)\equiv 0$, then

$$\hat{C}(t) = \int_0^t \frac{dA_0(s)}{\beta^0(s)}.$$

In addition, we have

$$\int_0^T 1_{\{\overline{V}(t)=0\}} h^2(t)dt = \int_0^T 1_{\{\overline{V}(t)=0\}}(\beta^\nu(t))^{-2} d < M_\nu > (t) = 0 \ \text{ a.s.}.$$

Following Karatzas and Kou [1], we obtain that the processes defined by

$$\hat{\pi}(t) \triangleq \sigma^{-1}(t)h(t), \quad \hat{p}(t) \triangleq \frac{\hat{\pi}(t)}{\overline{V}(t)} \cdot 1_{\{\overline{V}(t)>0\}}, \tag{5.20}$$

are **F**-progressively measurable and satisfy $\int_0^T \hat{\pi}^2(t)dt < \infty$, a.s.. While (5.18) implies that

$$h(t) = \frac{\psi_\nu(t)}{\beta^\nu(t)} = \overline{V}(t)\hat{p}(t)\sigma(t), \quad 0 \le t \le T. \tag{5.21}$$

Following [1], we also have

$$\hat{p}(t) \in K_+, \quad 0 \le t \le T. \tag{5.22}$$

Substitution of (5.21) back into (5.19) and (5.15) leads to

$$\begin{aligned}Q_\nu(t) = P + \int_0^t \beta^\nu(s)\overline{V}(s)\hat{p}(s)\sigma(s)dW^\nu(s) - \int_0^t \beta^\nu(s)d\hat{C}(s)\\ - \int_0^t \overline{V}(s)\beta^\nu(s)\Big(\delta(\nu_s) + \nu(s)\hat{p}(s)\Big)ds.\end{aligned} \tag{5.23}$$

Comparing with (5.10) we obtain

$$\beta^\nu(t)V^{P,\hat{\pi},\hat{C}}(t) = Q_\nu(t) = \beta^\nu(t\wedge\gamma_0)\overline{V}(t\wedge\gamma_0),$$

that is

$$V^{P,\hat{\pi},\hat{C}}(t) = \begin{cases} \overline{V}(t), & 0\le t<\gamma_0,\\ \overline{V}(\gamma_0)\frac{\beta^\nu(\gamma_0)}{\beta^\nu(t)}, & \gamma_0\le t\le T.\end{cases}$$

So for any $0\le t\le T$, we have

$$V^{P,\hat{\pi},\hat{C}}(t\wedge\gamma_0) \ge Z(t,\gamma_0),$$

thus we obtain $(\hat{\pi},\hat{C})\in\mathcal{A}_+(P,\gamma_0)$. □

Now let us consider the lower-hedging price of the GCC, let

$$\begin{aligned}&\underline{V}(t) = \operatorname*{essinf}_{\nu\in\mathcal{D}} V_\nu(t), \quad \tau_t \triangleq \inf\{u\in[t,T), \underline{V}(u) = X(u)\}\wedge T.\\ &p \triangleq \inf_{\nu\in\mathcal{D}} V_\nu(0) = \inf_{\nu\in\mathcal{D}}\sup_{\tau\in\mathcal{S}}\inf_{\gamma\in\mathcal{S}} \mathbf{E}^\nu[\beta^\nu(\tau\wedge\gamma)Z(\tau,\gamma)],\end{aligned} \tag{5.24}$$

Similarly, we can obtain the following results and the proof will be omitted.

Lemma 5.2. *Suppose that $P<\infty$, then $\mathbf{F}$-adapted process $\beta^\nu(\tau_0\wedge t)\underline{V}(\tau_0\wedge t)$ can be considered in its RCLL modification and the process $\{\beta^\nu(\tau_0\wedge t)\underline{V}(\tau_0\wedge t)\}_{0\le t\le T}$ is a $\mathbf{P}^\nu$-submartingale.*

Theorem 5.2. *The lower-hedging price $h_{low}(K)$ satisfies*

$$h_{low}(K) \le p = \inf_{\nu\in\mathcal{D}}\sup_{\tau\in\mathcal{S}}\inf_{\gamma\in\mathcal{S}} \mathbf{E}^\nu[\,\beta^\nu(\tau\wedge\gamma)Z(\tau,\gamma)], \tag{5.25}$$

with equality if $P<\infty$ or $p=0$. In the case that $P<\infty$, there exists a pair $(\check{\pi},\check{C})\in\mathcal{A}_-(-p,\tau_0)$, such that

$$V^{-p,\check{\pi},\check{C}}(\tau_0\wedge t) + Z(\tau_0,t) \ge 0, \quad 0\le t\le T. \tag{5.26}$$

We give a simple example as follows. Consider Israeli δ-penalty put options, that is

$$X(t) = (S(t)-q)^+, \quad Y(t) = (S(t)-q)^+ + \delta,$$

where $\delta>0$ is a constant. For the sake of plainness, we let $V_\nu(0)\triangleq V_\nu(x)=V_\nu(x,0)$. Denoting the price of the GCC starts from $S(0)=x$ in the market $\mathcal{M}_\nu$, we have

Proposition 5.1. *Suppose that $r(\cdot)\geq 0$, and that $0\leq\ell<\infty$ satisfies*

$$-\ell \leq \delta(y) + y \leq 0, \quad \forall\, y \in \widetilde{K}. \tag{5.27}$$

If $V_\nu(q)\leq\delta$, then

$$h_{up}(K) = \sup_{\nu\in\mathcal{D}} V_\nu^E(x), \quad h_{low}(K) = \inf_{\nu\in\mathcal{D}} V_\nu^E(x),$$

where

$$V_\nu^E(x) = V_\nu^E(x,0) = \mathbf{E}^\nu[\beta^\nu(T)(S(T)-q)^+].$$

If $V_\nu(q)>\delta$, then

$$h_{up}(K) \leq \sup_{\nu\in\mathcal{D}} V_\nu^E(x), \quad h_{low}(K) \leq \inf_{\nu\in\mathcal{D}} V_\nu^E(x).$$

Proof. As pointed out in [1] that from (5.27) we can obtain $P<\infty$. Then from Proposition 5.20 of [1], Theorem 2.2 (set $s=0$) of [4], and Theorems 5.1 and 5.2 we can complete the proof of the proposition. □

Proposition 5.2. *Suppose that the interest-rate $r(\cdot)$ satisfies $r(\cdot)\leq r$ for some constant $r\geq 0$, and that the function*

$$\delta(y) + y \geq 0, \quad \forall\, y \in \widetilde{K}. \tag{5.28}$$

Then we have

$$h_{up}(K) \leq S(0), \quad h_{low}(K) = X(0) = (S(0)-q)^+.$$

Proof. If $\delta\geq V_\nu(q)$, then it is not optimal for the writer to terminate the contract in advance. From (5.28) we know that $\beta_\nu(\tau)S(\tau)$ is a $\mathbf{P}^\nu$-supermartingale, so

$$\begin{aligned} V_\nu(x) = V_\nu^A(x) &\triangleq \sup_{\tau\in\mathcal{S}} \mathbf{E}^\nu[\beta_\nu(\tau)X(\tau)] \\ = \mathbf{E}^\nu[\beta_\nu(\tau)(S_\tau - q)^+] &\leq \mathbf{E}^\nu[\beta_\nu(\tau)S(\tau)] \leq \beta_\nu(0)S(0) = S(0). \end{aligned}$$

Consequently,

$$h_{up}(K) = \sup_{\nu\in\mathcal{D}} V_\nu(x) \leq S(0).$$

Since $V_\nu(x)=V_\nu^A(x)$, from [1] we can easily obtain

$$h_{low}(K) \leq (S_0 - q)^+ = X(0).$$

Moreover, $h_{low}(K)\geq X(0)$ has been mentioned earlier ($X(0)\in B$). So we have $h_{low}(K)=X(0)$.

If $\delta<V_\nu(q)$, then we have $V_\nu(x)\leq V_\nu^A(x)$. The desired result can be obtained in a similar way. □

6 Perpetual Israeli $\delta-$penalty put option with dividends

In this section, we will consider perpetual Israeli δ-penalty call options with constant coefficients $r>0$, $\sigma>0$, $q>0$, and one stock which pays dividends at a certain fixed rate $d\in(0,r)$. Here $0\leq t<\infty$. Denoting by $x=S(0)\in(0,\infty)$ the initial stock-price, we have

$$\begin{aligned} S(t) &= x\cdot\exp\{(r-d-\frac{1}{2}\sigma^2)t+\sigma W(t)\} \\ &= x\cdot\exp\{\sigma(W(t)-\rho t)\}, \qquad 0\leq t<\infty, \end{aligned} \tag{6.1}$$

where $\rho\triangleq(d-r)/\sigma+(\sigma)/2$. From [1], for $0<\epsilon<2\beta/\sigma^2$, we have

$$\mathbf{E}^0\Big[\sup_{0\leq t<\infty}\Big(e^{-rt}Y(t)\Big)^{1+\epsilon}\Big]<\infty, \quad \text{and} \quad \lim_{t\to\infty}e^{-rt}Y(t)=0, \quad a.s., \tag{6.2}$$

which implies that Theorem 3.1 holds.

First, we will consider the unconstrained case. Game option is introduced by kifer, Kyprianou and Ekström extend the research, for some special game options, obtain closed-form expressions of the prices through martingale tools and excessive functions. Here we shall sketch the main steps of solving this problem by means of variational arguments. First, let

$$G(x)\triangleq\sup_{\tau\in\mathcal{S}_{0,\infty}}\inf_{\gamma\in\mathcal{S}_{0,\infty}}\mathbf{E}^0\left[e^{-r(\tau\wedge\gamma)}Z(\tau,\gamma)\right], \quad 0<x<\infty. \tag{6.3}$$

Consequently,

$$V_0(t)=\operatorname*{ess\,sup}_{\tau\in\mathcal{S}_{t,\infty}}\ \operatorname*{ess\,inf}_{\gamma\in\mathcal{S}_{t,\infty}}\mathbf{E}^0\left[e^{-r(\tau\wedge\gamma-t)}Z(\tau,\gamma)|\mathcal{F}_t\right]=G(S_t). \tag{6.4}$$

Clearly, qua penalty, the size of δ determines the choice of stopping time for the writer. Beyond a certain value of δ it would not seem optimal for the writer to exercise at all. In order to depict this measurement, we introduce reward function as

$$g(x)\triangleq\sup_{\tau\in\mathcal{S}_{0,\infty}}\mathbf{E}^0\left[e^{-r\tau}(S_\tau-q)^+\right], \quad 0<x<\infty. \tag{6.5}$$

That is, the price of standard American call options, and the result is referred to Karatzas. Then when $\delta\geq g(q)$, it is not optimal for the writer to terminate the contract in advance, and we have $g(x)=G(x)$. When $\delta<g(q)$, experience indicates that the optimal stopping time for the writer is given by

$$\gamma_q=\inf\{t\geq 0: S(t)=q\}. \tag{6.6}$$

While the optimal stopping time for the holder should be searched in the form of

$$\tau_a=\inf\{t\geq 0: S(t)\geq a\}, \quad a\in(q,\infty). \tag{6.7}$$

In order to compute the optimal reward function of (6.3), we consider the problem

$$\begin{cases} G(x) = x - q, & x \in [a, \infty), \\ (\mathcal{L} - r)G(x) = 0, & x \in (0, q) \cup (q, a), \end{cases} \tag{6.8}$$

where

$$\mathcal{L} = (r - d)x\frac{\partial}{\partial x} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2}{\partial x^2}, \tag{6.9}$$

is the infinitesimal generator of the process S under measure $\mathbf{P}$. In addition, we introduce boundary conditions as

$$\lim_{x \to q} G(x) = \delta, \quad \lim_{x \uparrow a} G(x) = a - q, \quad \lim_{x \uparrow a} G'(x) = 1, \quad \lim_{x \downarrow 0} G(x) = 0. \tag{6.10}$$

We will omit the process of solving this equation and finally have

$$G(x) = \begin{cases} Ax^{\beta_1}, & 0 < x \leq q, \\ Bx^{\beta_1} + Cx^{\beta^2}, & q < x < a, \\ x - q, & a \leq x < \infty, \end{cases} \tag{6.11}$$

where

$$A = \delta q^{-\beta_1}, \quad B = \frac{(a-q)q^{\beta_2} - \delta a^{\beta^2}}{q^{\beta_2}a^{\beta_1} - q^{\beta_1}a^{\beta_2}}, \quad C = \frac{\delta a^{\beta^1} - (a-q)q^{\beta_1}}{q^{\beta_2}a^{\beta_1} - q^{\beta_1}a^{\beta_2}}. \tag{6.12}$$

While $q{=}ay$, y is the solution in $(0, 1)$ to the equation

$$y^{\beta_2} - y^{\beta_1} + \frac{\delta}{q}(\beta_1 - \beta_2)y - (1 - y)(\beta_1 y^{\beta_2} - \beta_2 y^{\beta_1}) = 0, \tag{6.13}$$

where

$$\beta_1 = \frac{\rho + \sqrt{\rho^2 + 2r}}{\sigma} \in (1, \frac{r}{r-d}), \quad \beta_2 = \frac{\rho - \sqrt{\rho^2 + 2r}}{\sigma} < 0, \tag{6.14}$$

are the roots of the equation

$$\frac{\sigma}{2}\beta^2 - \rho\beta - \frac{r}{\sigma} = 0.$$

Finally, we need the convexity of $G(x)$ to validate that it is the price of this game option. From its expression, we only need to point out that it is convex at the point q, that is, $G'(q+) \geq G'(q-)$, which is equivalent to prove $y \geq (\beta_1 - 1)/\beta_1$. Note that if

$$\delta = g(q), \quad y = (\beta_1 - 1)/\beta_1,$$

when $\delta{=}0$, $y{=}1$. Thus, from (6.12) and (6.13) we can obtain the desired result. In addition, we get that optimal stopping times for the writer and holder are given by

$\tau^* \triangleq \tau_a$, $\gamma^* \triangleq \gamma_q$ respectively. Now let us consider the choice of the portfolio for both sides. For that, let $Y(t) = e^{-rt}G(S_t)$, and Ito formula gives

$$\begin{aligned} Y(t) &= Y(0) + \int_0^t e^{-ru}(\mathcal{L} - r)G(S_u)du + \int_0^t \sigma Y(u)\frac{S_u G'(S_u)}{G(S_u)}dW^0(u) \\ &\quad + \int_0^t e^{-ru}\Big(G'(q+) - G'(q-)\Big)dL(u) \\ &= P_0 + \int_0^t e^{-ru}(-d \cdot S_u + rq)1_{(S_u>a)}du + \int_0^t \sigma Y(u)\frac{S_u G'(S_u)}{G(S_u)}dW^0(u) \\ &\quad + \int_0^t e^{-ru}\Big(G'(q+) - G'(q-)\Big)dL(u), \end{aligned} \tag{6.15}$$

where $L(t)$ is the local time of $S(t)$ at q. Set $t{=}t \wedge \gamma^*$, then the last term of the above is 0 since the local time $L(t)$ increases only at t such that $S(t){=}q$. Therefore we have

$$\begin{aligned} Y(t \wedge \gamma^*) = P_0 &+ \int_0^{t\wedge\gamma^*} e^{-ru}(-d \cdot S_u + rq)1_{(S_u>a)}du \\ &+ \int_0^{t\wedge\gamma^*} \sigma Y(u)\frac{S_u G'(S_u)}{G(S_u)}dW^0(u). \end{aligned} \tag{6.16}$$

If let

$$\hat{C}(t) \triangleq \int_0^{t\wedge\gamma^*}(d \cdot S_u - rq)1_{(S_u>a)}du, \quad \hat{p}(t) \triangleq \frac{S_t G'(S_t)}{G(S_t)},$$

then the comparison with (5.10) (set $\nu \equiv 0$) gives

$$V^{P_0,\hat{\pi},\hat{C}}(t \wedge \gamma^*) = G(S_{t\wedge\gamma^*}) \geq Z(t, \gamma^*).$$

Similarly, for (6.15), set $t = \tau^* \wedge t$, then we have

$$Y(\tau^* \wedge t) = P_0 + \int_0^{\tau^*\wedge t} \sigma Y(u)\frac{S_u G'(S_u)}{G(S_u)}dW^0(u) + \int_0^{\tau^*\wedge t} e^{-ru}\Big(G'(q+) - G'(q-)\Big)dL(u).$$

Let

$$\check{C}(t) \triangleq \int_0^{\tau^*\wedge t}[G'(q+) - G'(q-)]dL(u), \quad \check{p}\,(t) \triangleq -\frac{S_t G'(S_t)}{G(S_t)}.$$

Consequently,

$$-V^{-P_0,\check{\pi},\check{C}}(\tau^* \wedge t) = G(S_{\tau^*\wedge t}) \leq Z(\tau^*, t).$$

For the sake of plainness, we summarize the above result as the following theorem.

Theorem 6.1. *The expression of the optimal reward function $G(x)$ is given by* (6.11)*, and the optimal portfolio $(\hat{\gamma}, \hat{\pi}, \hat{C})$ of the writer is given by*

$$\begin{aligned} &\hat{\gamma} = \inf\{t \geq 0 : S(t) = q\}, \quad \hat{p}(t) = \frac{S_t G'(S_t)}{G(S_t)}, \\ &\hat{C}(t) = \int_0^{t\wedge\gamma^*}(d \cdot S_u - rq)1_{(S_u>a)}du. \end{aligned} \tag{6.17}$$

From the definition of $\hat{p}(t)$ *we can obtain* $\hat{\pi}(t)$, *while the optimal hedging portfolio* $(\check{\tau}, \check{\pi}, \check{C})$ *of the holder is given by*

$$\check{\tau} = \inf\{t \geq 0 : S(t) \geq a\}, \quad \check{\pi}(t) = -\hat{\pi}(t),$$
$$\check{C}(t) = \int_0^{\tau^* \wedge t} [G'(q+) - G'(q-)] dL(u). \tag{6.18}$$

Now, let us deal with closed constraints on portfolio. Clearly, under the case with dividend we have

$$dS(t) = S(t)\left[(r - d - \nu(t))dt + \sigma dW^\nu(t)\right], \quad 0 \leq t < \infty,$$
$$\beta^\nu(t)S(t) = \exp\left\{-\int_0^t [\delta(\nu_s) + \nu(s) + d]ds + \sigma W^\nu(t) - \frac{1}{2}\sigma^2 t\right\}. \tag{6.19}$$

From the proof of Proposition 5.2, we know that it remains valid in this infinite-horizon case.

Example 6.1. Prohibition of short-selling of stock. That is, the capital invested in the stock can not be negative. Then we have

$$K_+ = [0, \infty), \quad K_- = (-\infty, 0], \quad \delta(y) = 0, \quad \widetilde{K} = [0, \infty).$$

Thus, for any $y \in \widetilde{K}$, $\delta(y) + y = y$, it satisfies the condition of Proposition 5.2, so we have

$$h_{low}(K) = X(0) = (S(0) - q)^+, \quad h_{up}(K) = P(0) = G(S(0)) \leq S(0). \tag{6.20}$$

Indeed, the first claim can be obtained by Proposition 5.2 directly. For the second claim, as mentioned in section 4 that $h_{up}(K) \geq P(0)$ holds, we only need to give the reverse inequality. From (6.17) we know $\hat{p}(\cdot) \in K_+$, and the triple $(\hat{\gamma}, \hat{\pi}, \hat{C})$ given by Theorem 6.1 satisfies

$$V^{P_0, \hat{\pi}, \hat{C}}(t \wedge \hat{\gamma}) \geq Z(t, \hat{\gamma}). \tag{6.21}$$

So $P(0) \in A$. It follows from the definition of $h_{up}(K)$ that $h_{up}(K) \leq P(0)$.

Example 6.2. Constraints on the short-selling of stock. That is, the wealth invested in the stock can be negative; however, there must have some limit. Then in this case we have

$$K_+ = [-k, \infty), \quad K_- = (-\infty, -k], \quad \delta(y) = ky, \quad \widetilde{K} = [0, \infty),$$

where $k > 0$ is a constant. Thus, for any $y \in \widetilde{K}$, $\delta(y) + y = (1 + k)y$, which satisfies the condition of Proposition 5.2. Similar to Example 6.1, we can also show that (6.20) holds.

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