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PARALLEL CHAOTIC MULTISPLITTING ITERATIVE METHODS FOR THE LARGE SPARSE LINEAR COMPLEMENTARITY PROBLEM[*1)]

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Abstract

A parallel chaotic multisplitting method for solving the large sparse linear complementarity problem is presented, and its convergence properties are discussed in detail when the system matrix is either symmetric or nonsymmetric. Moreover, some applicable relaxed variants of this parallel chaotic multisplitting method together with their convergence properties are investigated. Numerical results show that highly parallel efficiency can be achieved by these new parallel chaotic multisplitting methods.

Key words: Linear complementarity problem, Matrix multisplitting, Chaotic iteration, Relaxed method, Convergence property.

1. Introduction

We consider the linear complementarity problem LCP(M,q): Find a $z \in \mathbb{R}^n$ such that

$$Mz + q \geq 0, \qquad z \geq 0, \qquad z^T(Mz+q) = 0,$$

where $M = (m_{ij}) \in \mathbb{R}^{n\times n}$ and $q = (q_i) \in \mathbb{R}^n$ are given real matrix and vector, respectively. This problem arises in various scientific computing areas such as the Nash equilibrium point of a bimatrix game (e.g., Cottle and Dantzig[4] and Lemke[12]) and the free boundary problems of fluid mechanics (e.g., Cryer[8]). There have been a lot of researches on the approximate solution of the linear complementarity problem LCP(M,q). For details one can refer to Cottle, Pang and Stone[6] and references therein. These researches presented efficient iterative methods and systematic convergence theories for solving the linear complementarity problem in the sequential computing environment.

To solve the linear complementarity problem in the parallel computing environment, Machida, Fukushima and Ibaraki[14] recently presented a multisplitting iterative method by making use of the matrix multisplitting technique introduced in O'Leary and White[17]. Under suitable conditions about the weighting matrices and the multiple splittings, Machida, Fukushima and Ibaraki[14] and Bai[1] proved the convergence of this method for symmetric and nonsymmetric linear complementarity problems, respectively. This method possesses good parallel computational properties, and it is much suitable for implementing on the synchronous parallel multiprocessor systems. It can achieve high parallel efficiency provided the workloads among the processors of the multiprocessor system are well balanced. When such a balance can be obtained, the individual processor is then ready to contribute towards their update of the global iterate almost at the same time, which, in turn, minimizes idle time. However, such a balance

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of workload is not always available in many applications, and the mutual wait among the processors of the multiprocessor system is usually inevitable, which, hence, decreases the parallel efficiency of the multisplitting method.

To avoid loss of time and efficiency in processor utilization, in this paper, we propose a chaotic multisplitting iterative method for solving parallely the linear complementarity problem LCP(M,q). In the implementation of this method on the multiprocessor system, each processor can carry out its local iterate a varying number of steps until a mutual phase time is reached when all processors are ready to contribute towards the global iteration. Hence, the synchronous wait among different processors is greatly decreased while the efficient numerical computation on each processor is largely increased. This, therefore, makes the new chaotic multisplitting method achieve high parallel efficiency. Under the same restrictions on the weighting matrices and the multiple splittings as in [14] and [1], we establish the convergence theories of this new method for both the symmetric and nonsymmetric linear complementarity problems. Moreover, for the convenience of practical implementations, some relaxed explicit variants of the above chaotic multisplitting method are presented, and their convergence for both the symmetric and nonsymmetric linear complementarity problems are discussed in detail as well. At last, with a lot of numerical results, we show that the new chaotic multisplitting methods are feasible and efficient for parallely solving the linear complementarity problems on the multiprocessor systems.

2. Preliminaries

First of all, we briefly review some necessary notations and concepts in [1] and [14]. A matrix $A=(a_{ij})\in\mathbb{R}^{n\times n}$ is called a monotone matrix if it is nonsingular and satisfies $A^{-1}\geq 0$; an M-matrix if it is a monotone matrix and satisfies $a_{ij}\leq 0$ for $i\neq j$, $i,j=1,2,\cdots,n$; an H-matrix if its comparison matrix $\langle A\rangle$ is an M-matrix, where $\langle A\rangle=(\langle a_{ij}\rangle)\in\mathbb{R}^{n\times n}$ is defined by $\langle a_{ii}\rangle=|a_{ii}|$ for $i=1,2,\cdots,n$, and $\langle a_{ij}\rangle=-|a_{ij}|$ for $i\neq j$, $i,j=1,2,\cdots,n$; an H_+-matrix if it is an H-matrix having positive diagonal elements; and a Q-matrix if the LCP(A,b) has a solution for any $b\in\mathbb{R}^n$. A sufficient condition for $A\in\mathbb{R}^{n\times n}$ to be a Q-matrix is that either A is an H_+-matrix[1] or A is a strictly copositive matrix [6]. In the former case, the LCP(A,b) always has a unique solution for every $b\in\mathbb{R}^n$. For a given matrix $A\in\mathbb{R}^{n\times n}$, let $F,G\in\mathbb{R}^{n\times n}$ be such that $A=F+G$. Then (F,G) is called a splitting of the matrix A. The splitting (F,G) is called a convergent splitting if the spectral radius of the matrix $(F^{-1}G)$ is less than one, i.e., $\rho(F^{-1}G)<1$. It is called an M-splitting if F is an M-matrix and $G\leq 0$; an H-splitting if $\langle F\rangle-|G|$ is an M-matrix; an H-compatible splitting if $\langle A\rangle=\langle F\rangle-|G|$; and a Q-splitting if F is a Q-matrix. In particular, the splitting (F,G) is called an H_+-splitting and H_+-compatible splitting if it is an H-splitting and H-compatible splitting, respectively, with F an H_+-matrix. Let $N_0=\{0,1,2,\cdots\}$ and $\{A_p\}_{p\in N_0}$ be a sequence of matrices in $\mathbb{R}^{n\times n}$. Then we call $A_p(p\in N_0)$ positive definite uniformly in p if there exists a positive constant c, independent of p, such that $z^TA_pz\geq cz^Tz$ holds for all $z\in\mathbb{R}^n$.

The following lemmas, proved in [20] and [9] respectively, will frequently be used in the sequel.

Lemma 2.1. [20] *Let $A\in\mathbb{R}^{n\times n}$ have nonpositive off-diagonal entries. Then A is an M-matrix if and only if there exists a positive vector $u\in\mathbb{R}^n$ such that $Au\geq 0$.*

Lemma 2.2. [9] *Let $A\in\mathbb{R}^{n\times n}$ be an H-matrix, $D=diag(A)$, and $A=D-B$. Then:*

(a) *A is nonsingular;*

(b) *$|A^{-1}|\leq\langle A\rangle^{-1}$; and*

(c) *$|D|$ is nonsingular and $\rho(|D|^{-1}|B|)<1$.*

When the system matrix $M\in\mathbb{R}^{n\times n}$ is symmetric, associated with the LCP(M,q) is the following quadratic programming problem QP(M,q):

$$\begin{cases} \text{minimize} \quad f(z) = \frac{1}{2} z^T M z + q^T z, \\ \text{subject to} \quad z \geq 0. \end{cases}$$

It is easily seen that the QP(M,q) is a necessary optimality condition for the LCP(M,q), and it is also sufficient if $M \in \mathbb{R}^{n\times n}$ is positive semidefinite.

The next result is essential for proving the convergence of our new parallel chaotic multisplitting method for the symmetric linear complementarity problem.

Lemma 2.3. [6] *Let $M \in \mathbb{R}^{n\times n}$ be a symmetric matrix, and $M = B + C$ be a Q-splitting. Given $z \geq 0$ and let $\overline{z}$ be an arbitrary solution of the LCP(B,q(z)):*

$$\overline{z} \geq 0, \quad B\overline{z} + q(z) \geq 0, \quad \overline{z}^T(B\overline{z} + q(z)) = 0,$$

where $q(z) = Cz + q$. Then it holds that

$$f(z) - f(\overline{z}) \geq \frac{1}{2}(z - \overline{z})^T (B - C)(z - \overline{z}).$$

3. Parallel Chaotic Multisplitting Method

A multisplitting of a matrix $M \in \mathbb{R}^{n\times n}$ is a collection of triples $(B_k, C_k, E_k)(k = 1, 2, \cdots, K)$, which satisfies

(1) $M = B_k + C_k(k = 1, 2, \cdots, K)$ are Q-splittings; and

(2) $E_k(k = 1, 2, \cdots, K)$ are nonnegative diagonal matrices with $\sum_{k=1}^{K} E_k = I$ (the $n \times n$ identity matrix).

Let $(B_{k,p}, C_{k,p}, E_{k,p})(k = 1, 2, \cdots, K)$, $p \in N_0$, be a sequence of multisplittings of the matrix M. Then we consider the following parallel chaotic multisplitting method for solving the LCP(M,q).

Method 3.1. (Parallel Chaotic Multisplitting Method for the LCP(M,q))

Step 1. Choose an arbitrary starting vector $z^0 \in \mathbb{R}^n$. Set $p := 0$.

Step 2. For each $k \in \{1, 2, \cdots, K\}$, set $z^{k,p,0} := z^p$, and take a positive integer $m_{k,p}$.

Step 3. For each $k \in \{1, 2, \cdots, K\}$ and $m = 1$ to $m_{k,p}$, let $z^{k,p,m}$ be an arbitrary solution of the LCP$(B_{k,p}, q_{k,p,m})$:

$$z \geq 0, \qquad B_{k,p}z + q_{k,p,m} \geq 0, \qquad z^T(B_{k,p}z + q_{k,p,m}) = 0,$$

where $q_{k,p,m} = C_{k,p}z^{k,p,m-1} + q$.

Step 4. For each $k \in \{1, 2, \cdots, K\}$, set $z^{k,p+1} := z^{k,p,m_{k,p}}$.

Step 5. $z^{p+1} = \sum_{k=1}^{K} E_{k,p} z^{k,p+1}$.

Step 6. If $z^{p+1} = z^p$, then stop. Otherwise, set $p := p + 1$ and return to Step 2.

At every iterate step p of Method 3.1, each sub-problem LCP$(B_{k,p}, \bullet)$ $(k = 1, 2, \cdots, K)$ is solved independently on one processor of the multiprocessor system. Moreover, each processor can carry out its local iteration a varying number $m_{k,p}$ of steps until a mutual phase time is reached when all processors are ready to contribute towards the global iteration. Hence, synchronous wait and idle time are minimized, which makes Method 3.1 achieve high parallel efficiency. On the other hand, considerable savings on the computational workload are available because the entries of $z^{k,p+1}$ corresponding to the zero elements of the weighting matrix $E_{k,p}$ need not be computed.

The composition numbers $m_{k,p}$ can be determined either precedently or dynamically. If $m_{k,p} = 1$ $(k = 1, 2, \cdots, K, p \in N_0)$, Method 3.1 reduces to the multisplitting method for the LCP(M,q) discussed in Machida, Fukushima and Ibaraki[14] and Bai[1]. We remark that Method 3.1 is also a generalization and development of Model A in Bru, Elsner and Neumann[3] for the systems of linear equations to the linear complementarity problems.

4. Convergence Theories

In this section, we will discuss the convergence of Method 3.1 for the cases that the system matrix $M \in \mathbb{R}^{n\times n}$ belongs to the symmetric and nonsymmetric matrix classes, respectively. First of all, we establish the convergence theorem for the symmetric case.

Theorem 4.1. *Let $M \in \mathbb{R}^{n\times n}$ be a symmetric matrix, and for every $p \in N_0$, $(B_{k,p}, C_{k,p}, E_{k,p})$ $(k = 1, 2, \cdots, K)$ be a multisplitting of the matrix M such that*

(a) *$B_{k,p} - C_{k,p} (k = 1, 2, \cdots, K)$ are positive definite uniformly in p; and*

(b) *$f(\sum_{k=1}^{K} E_{k,p} z^{k,p}) \le \max_{1\le k\le K} f(z^{k,p})$.*

Let $m_{k,p} (k = 1, 2, \cdots, K, p \in N_0)$ be positive integers bounded uniformly from above, and assume that $f(z)$ is bounded below on $z \ge 0$, and that $0 \ne z \ge 0$, $Mz \ge 0$ and $z^T M z = 0$ imply $q^T z > 0$. Then, independently of the positive integer sequences $\{m_{k,p}\}_{p\in N_0}$ $(k = 1, 2, \cdots, K)$, the sequence $\{z^p\}_{p\in N_0}$ generated by Method 3.1 is bounded and any accumulation point of it solves the LCP(M,q).

Proof. By Lemma 2.3 we have

$$f(z^{k,p,m-1}) - f(z^{k,p,m}) \ge \frac{1}{2}(z^{k,p,m-1} - z^{k,p,m})^T (B_{k,p} - C_{k,p})(z^{k,p,m-1} - z^{k,p,m}),$$
$$m = 1, 2, \cdots, m_{k,p}; \quad p \in N_0.$$

In accordance with assumption (a) there exists a positive constant c, independent of p and k, such that

$$z^T (B_{k,p} - C_{k,p}) z \ge c z^T z, \qquad k = 1, 2, \cdots, K, \quad p \in N_0, \quad \forall z \in \mathbb{R}^n.$$

Therefore,

$$\begin{aligned} f(z^p) - f(z^{k,p,m_{k,p}}) &= \sum_{m=1}^{m_{k,p}} \left(f(z^{k,p,m-1}) - f(z^{k,p,m}) \right) \\ &\ge \frac{1}{2} \sum_{m=1}^{m_{k,p}} (z^{k,p,m-1} - z^{k,p,m})^T (B_{k,p} - C_{k,p})(z^{k,p,m-1} - z^{k,p,m}) \qquad (4.1) \\ &\ge \frac{c}{2} \sum_{m=1}^{m_{k,p}} \|z^{k,p,m-1} - z^{k,p,m}\|_2^2. \end{aligned}$$

In addition, by further noticing that $m_{k,p} (k = 1, 2, \cdots, K, p \in N_0)$ are uniformly bounded from above by a positive integer, say J, we can get

$$
\begin{aligned}
\|z^p - z^{k,p,m_{k,p}}\|_2^2 &= \left\| \sum_{m=1}^{m_{k,p}} \left(z^{k,p,m-1} - z^{k,p,m}\right) \right\|_2^2 \\
&\le \left(\sum_{m=1}^{m_{k,p}} \|z^{k,p,m-1} - z^{k,p,m}\|_2 \right)^2 \\
&\le m_{k,p} \sum_{m=1}^{m_{k,p}} \|z^{k,p,m-1} - z^{k,p,m}\|_2^2 \\
&\le J \sum_{m=1}^{m_{k,p}} \|z^{k,p,m-1} - z^{k,p,m}\|_2^2.
\end{aligned}
$$

Substituting this estimate into (4.1) yields

$$
f(z^p) - f(z^{k,p+1}) \ge \frac{c}{2J} \|z^p - z^{k,p+1}\|_2^2, \qquad k = 1, 2, \cdots, K, \quad p \in N_0. \tag{4.2}
$$

From assumption (b) and (4.2) we know that

$$
\begin{aligned}
f(z^p) - f(z^{p+1}) &= f(z^p) - f(\sum_{k=1}^{K} E_{k,p} z^{k,p+1}) \\
&\ge f(z^p) - \max_{1\le k\le K} f(z^{k,p+1}) \\
&= f(z^p) - f(z^{k_{p+1},p+1}) \\
&\ge \frac{c}{2J} \|z^p - z^{k_{p+1},p+1}\|_2^2
\end{aligned} \tag{4.3}
$$

holds for all $p \in N_0$, where k_{p+1} is an index such that $f(z^{k_{p+1},p+1}) = \max_{1\le k\le K} f(z^{k,p+1})$.

We first prove that the sequence $\{z^p\}_{p\in N_0}$ is bounded. Otherwise, suppose that the sequence $\{z^p\}_{p\in N_0}$ is unbounded. Then there exists at least one subsequence $\{z^{p_\ell}\}_{\ell\in N_0}$ such that $\|z^{p_\ell}\|_2 \to \infty$ as $\ell \to \infty$. Let the positive integer $k_{p+1} \in \{1, 2, \cdots, K\}$ be defined as in (4.3). Then by taking a further subsequence if necessary, we may assume that there exists some index $\hat{k} \in \{1, 2, \cdots, K\}$ such that $k_{p_\ell+1} = \hat{k}$ for all $\ell \in N_0$. Since (4.3) implies that the sequence $\{f(z^{p_\ell})\}_{\ell\in N_0}$ is convergent, the sequence $\{z^{p_\ell} - z^{\hat{k},p_\ell+1}\}_{\ell\in N_0}$ converges to zero as $\ell \to \infty$. This, in particular, shows that $\|z^{\hat{k},p_\ell+1}\|_2 \to \infty(\ell \to \infty)$ because $\|z^{p_\ell}\|_2 \to \infty(\ell \to \infty)$.

Now, consider the corresponding normalized sequence $\left\{\frac{z^{\hat{k},p_\ell+1}}{\|z^{\hat{k},p_\ell+1}\|_2}\right\}_{\ell\in N_0}$, which is bounded and hence has an accumulation point $\widetilde{z}$ such that $\|\widetilde{z}\|_2 = 1$ and $\widetilde{z} \ge 0$. Assume, without loss of generality, that $\left\{\frac{z^{\hat{k},p_\ell+1}}{\|z^{\hat{k},p_\ell+1}\|_2}\right\}_{\ell\in N_0}$ converges to $\widetilde{z}$. Then since $(z^{\hat{k},p_\ell+1} - z^{p_\ell}) \to 0(\ell \to \infty)$ and $\frac{z^{\hat{k},p_\ell+1}}{\|z^{\hat{k},p_\ell+1}\|_2} \to \widetilde{z}(\ell \to \infty)$, we have $\frac{z^{p_\ell}}{\|z^{\hat{k},p_\ell+1}\|_2} \to \widetilde{z}(\ell \to \infty)$. From (4.1) it holds that $\{z^{\hat{k},p_\ell,m} - z^{p_\ell}\}_{\ell\in N_0}$ converges to zero for any $m \in \{1, 2, \cdots, m_{\hat{k},p_\ell}\}$. Moreover, since $z^{\hat{k},p_\ell+1}$ is a solution of the LCP$(B_{\hat{k},p_\ell}, q_{\hat{k},p_\ell,m_{\hat{k},p_\ell}})$, we have

$$
B_{\hat{k},p_\ell} z^{\hat{k},p_\ell+1} + C_{\hat{k},p_\ell} z^{\hat{k},p_\ell,m_{\hat{k},p_\ell}-1} + q \ge 0, \tag{4.4a}
$$

$$
z^{\hat{k},p_\ell+1} \ge 0, \tag{4.4b}
$$

$$
\left(z^{\hat{k},p_\ell+1}\right)^T \left(B_{\hat{k},p_\ell} z^{\hat{k},p_\ell+1} + C_{\hat{k},p_\ell} z^{\hat{k},p_\ell,m_{\hat{k},p_\ell}-1} + q\right) = 0. \tag{4.4c}
$$

Then passing to the limit $\ell \to \infty$, from (4.4) we immediately get

$$\widetilde{z} \geq 0, \quad M\widetilde{z} \geq 0, \quad \widetilde{z}^T M\widetilde{z} = 0. \tag{4.5}$$

On the other hand, if the assumption that $f(z)$ is bounded below on $z \geq 0$ is satisfied, then $M \in \mathbb{R}^{n\times n}$ satisfies the inequality $z^T Mz \geq 0$ for any $z \geq 0$, by Proposition 7.3.14 in [6]. Because of $M = B_{\hat{k},p_\ell} + C_{\hat{k},p_\ell}$ and $z^{\hat{k},p_\ell+1} \geq 0$ by (4.4b), we see from (4.4c) that

$$\left(z^{\hat{k},p_\ell+1}\right)^T Mz^{\hat{k},p_\ell+1} = -\left(z^{\hat{k},p_\ell+1}\right)^T \left(C_{\hat{k},p_\ell}\left(z^{p_\ell} - z^{\hat{k},p_\ell+1}\right) + q\right) \geq 0. \tag{4.6}$$

Recalling that $(z^{\hat{k},p_\ell+1} - z^{p_\ell}) \to 0 (\ell \to \infty)$, dividing (4.6) by $\|z^{\hat{k},p_\ell+1}\|_2$ and then passing to the limit $\ell \to \infty$, we obtain the inequality $q^T\widetilde{z} \leq 0$. However, this together with (4.5) contradicts the assumption that $0 \neq z \geq 0$, $Mz \geq 0$ and $z^T Mz = 0$ imply $q^T z > 0$. Therefore, the sequence $\{z^p\}_{p\in N_0}$ must be bounded.

In the following, we will further demonstrate that every accumulation point of the sequence $\{z^p\}_{p\in N_0}$ is a solution of the LCP(M,q). To this end, we let $\widehat{z}$ be an arbitrary accumulation point of the sequence $\{z^p\}_{p\in N_0}$, and $\{z^{p_\ell}\}_{\ell\in N_0}$ be a subsequence that converges to $\widehat{z}$. Since $\{f(z^{p_\ell})\}_{\ell\in N_0}$ converges to $f(\widehat{z})$ as $\ell \to \infty$ and $\{f(z^p)\}_{p\in N_0}$ is nonincreasing by (4.3), the entire sequence $\{f(z^p)\}_{p\in N_0}$ converges to $f(\widehat{z})$, too. Let $\hat{k}$ be defined as before. Then the sequence $\{z^{p_\ell} - z^{\hat{k},p_\ell+1}\}_{\ell\in N_0}$ converges to zero by (4.3), and the sequence $\{z^{\hat{k},p_\ell+1}\}_{\ell\in N_0}$ converges to $\widehat{z}$ as $\ell \to \infty$. From (4.1) it further holds that for any $m \in \{1, 2, \cdots, m_{\hat{k},p_\ell}\}$, the sequence $\{z^{\hat{k},p_\ell,m}\}_{\ell\in N_0}$ converges to $\widehat{z}$ as $\ell \to \infty$. Because $z^{\hat{k},p_\ell+1}$ is a solution of the LCP($B_{\hat{k},p_\ell}, q_{\hat{k},p_\ell,m_{\hat{k},p_\ell}}$) defined by (4.4a)-(4.4c), it follows that $\widehat{z}$ solves the LCP(M,q).

Up to now, the proof of this theorem is fulfilled.

We remark that assumption (a) in Theorem 4.1 is a standard condition imposed to guarantee the convergence of the iterative methods for the LCP(M,q), and assumption (b) in Theorem 4.1 can be satisfied by various choices of the weighting matrices. For detailes, one can refer to [1] and [14]. In addition, the assumptions that $f(z)$ is bounded below on $z \geq 0$ and that $0 \neq z \geq 0$, $Mz \geq 0$ and $z^T Mz = 0$ imply $q^T z > 0$ in Theorem 4.1 are assured if $M \in \mathbb{R}^{n\times n}$ is a strictly copositive matrix [6, Theorem 5.3.5]. If $M \in \mathbb{R}^{n\times n}$ is a positive semidefinite matrix, they can be replaced by the assumption that there exists a $z \in \mathbb{R}^n$ such that $Mz + q > 0$ [6, Theorem 5.3.9].

Now, we turn to deal with the convergence of the nonsymmetric linear complementarity problem.

Theorem 4.2. *Let $M \in \mathbb{R}^{n\times n}$ be an H_+-matrix. Assume that for each $p \in N_0$ and $k \in \{1, 2, \cdots, K\}$, $M = B_{k,p} + C_{k,p}$ is an H_+-compatible splitting, satisfying $diag(B_{k,p}) \leq \beta\, diag(M)$ for some positive constant $\beta \geq 1$. Then the sequence $\{z^p\}_{p\in N_0}$ generated by Method 3.1 converges, independently of the positive integer sequences $\{m_{k,p}\}_{p\in N_0}$ $(k = 1, 2, \cdots, K)$, to the unique solution of the LCP(M,q).*

Proof. Because $M \in \mathbb{R}^{n\times n}$ is an H_+-matrix, the LCP(M,q) has a unique solution $z^* \in \mathbb{R}^n$. That is to say, it holds that

$$z^* \geq 0, \qquad Mz^* + q \geq 0, \qquad (z^*)^T(Mz^* + q) = 0. \tag{4.7}$$

On the other hand, since for each $k \in \{1, 2, \cdots, K\}$ and each $p \in N_0$, $B_{k,p}$ is an H_+-matrix, we know that the LCP($B_{k,p}, q_{k,p,m}$) has a unique solution $z^{k,p,m}$, where $q_{k,p,m} = C_{k,p}z^{k,p,m-1} + q$; i.e.,

$$z^{k,p,m} \geq 0, \qquad B_{k,p}z^{k,p,m} + q_{k,p,m} \geq 0, \qquad (z^{k,p,m})^T(B_{k,p}z^{k,p,m} + q_{k,p,m}) = 0. \tag{4.8}$$

Now, we claim that the following inequality holds:

$$|z^{k,p,m} - z^*| \leq \langle B_{k,p}\rangle^{-1}|C_{k,p}||z^{k,p,m-1} - z^*|, \qquad \forall p \in N_0. \tag{4.9}$$

In fact, by denoting $D_{k,p} = diag(B_{k,p})$ and $\widetilde{B}_{k,p} = B_{k,p} - D_{k,p}$, we can verify (4.9) for different cases.

First, when the i-th element of z^*, $[z^*]_i$, is positive and $[B_{k,p}z^{k,p,m} + q_{k,p,m}]_i = 0$, it clearly holds that

$$[B_{k,p}(z^{k,p,m} - z^*) + C_{k,p}(z^{k,p,m-1} - z^*)]_i = 0,$$

or equivalently,

$$[D_{k,p}(z^{k,p,m} - z^*)]_i = [-\tilde{B}_{k,p}(z^{k,p,m} - z^*) - C_{k,p}(z^{k,p,m-1} - z^*)]_i.$$

Therefore,

$$[D_{k,p}|z^{k,p,m} - z^*|]_i \leq [|\tilde{B}_{k,p}||z^{k,p,m} - z^*| + |C_{k,p}||z^{k,p,m-1} - z^*|]_i,$$

and hence

$$[\langle B_{k,p}\rangle|z^{k,p,m} - z^*|]_i \leq [|C_{k,p}||z^{k,p,m-1} - z^*|]_i. \tag{4.10}$$

Second, when $[z^*]_i > 0$ and $[B_{k,p}z^{k,p,m} + q_{k,p,m}]_i > 0$, it holds that $[z^{k,p,m}]_i = 0$ and

$$\begin{cases} [\widetilde{B}_{k,p}z^{k,p,m} + C_{k,p}z^{k,p,m-1} + q]_i > 0, \\ [\widetilde{B}_{k,p}z^* + C_{k,p}z^* + q]_i = -[D_{k,p}z^*]_i. \end{cases}$$

After subtracting these two inequalities we obtain

$$[\widetilde{B}_{k,p}(z^{k,p,m} - z^*) + C_{k,p}(z^{k,p,m-1} - z^*)]_i > [D_{k,p}z^*]_i.$$

Therefore,

$$\begin{aligned}[D_{k,p}|z^{k,p,m} - z^*|]_i = [D_{k,p}z^*]_i &< |[\widetilde{B}_{k,p}(z^{k,p,m} - z^*) + C_{k,p}(z^{k,p,m-1} - z^*)]_i| \\ &\leq [|\widetilde{B}_{k,p}||z^{k,p,m} - z^*| + |C_{k,p}||z^{k,p,m-1} - z^*|]_i,\end{aligned}$$

and hence, (4.10) also holds.

Finally, noticing that when $[z^*]_i = 0$ and $[z^{k,p,m}]_i = 0$, (4.10) holds automatically; and that when $[z^*]_i = 0$ and $[z^{k,p,m}]_i > 0$, (4.10) can be demonstrated analogously to the second case, we conclude that (4.10) holds for all $i \in \{1, 2, \cdots, n\}$, all $k \in \{1, 2, \cdots, K\}$ and all $p \in N_0$.

Since $B_{k,p}(k = 1, 2, \cdots, K, p \in N_0)$ are H_+-matrices, we know that (4.9) is valid.

(4.9) straightforwardly gives

$$|z^{k,p+1} - z^*| \leq \left(\langle B_{k,p}\rangle^{-1}|C_{k,p}|\right)^{m_{k,p}} |z^p - z^*|,$$

and hence,

$$\begin{aligned}|z^{p+1} - z^*| &= \left|\sum_{k=1}^{K} E_{k,p}(z^{k,p+1} - z^*)\right| \leq \sum_{k=1}^{K} E_{k,p}|z^{k,p+1} - z^*| \\ &\leq \sum_{k=1}^{K} E_{k,p}\left(\langle B_{k,p}\rangle^{-1}|C_{k,p}|\right)^{m_{k,p}} |z^p - z^*| := H_p|z^p - z^*|,\end{aligned} \tag{4.11}$$

where

$$H_p = \sum_{k=1}^{K} E_{k,p}\left(\langle B_{k,p}\rangle^{-1}|C_{k,p}|\right)^{m_{k,p}}. \tag{4.12}$$

Because $M \in \mathbb{R}^{n\times n}$ is an H_+-matrix, from Lemma 2.1 we know that there exists a positive vector $u \in \mathbb{R}^n$ such that $e = \langle M\rangle u$, where $e = (1, 1, \cdots, 1)^T \in \mathbb{R}^n$. On the other hand, that $M = B_{k,p} + C_{k,p}(k = 1, 2, \cdots, K, p \in N_0)$ are H_+-compatible splittings directly gives

$$\langle M\rangle = \langle B_{k,p}\rangle - |C_{k,p}|, \qquad k = 1, 2, \cdots, K, \quad p \in N_0.$$

Therefore,

$$\begin{aligned}\langle B_{k,p}\rangle^{-1}|C_{k,p}|u &= \langle B_{k,p}\rangle^{-1}(\langle B_{k,p}\rangle - \langle M\rangle)u \\ &= u - \langle B_{k,p}\rangle^{-1}e \\ &\le u - D_{k,p}^{-1}e \\ &\le u - \frac{1}{\beta}D^{-1}e,\end{aligned}$$

where $D = diag(M)$ is a positive diagonal matrix. This estimate clearly shows that there exists a constant $\theta \in [0, 1)$ such that

$$\langle B_{k,p}\rangle^{-1}|C_{k,p}|u \le \theta u, \qquad k = 1, 2, \cdots, K, \quad p \in N_0. \tag{4.13}$$

(4.12) and (4.13) together give the inequalities

$$H_p u = \sum_{k=1}^{K} E_{k,p}\left(\langle B_{k,p}\rangle^{-1}|C_{k,p}|\right)^{m_{k,p}} u \le \sum_{k=1}^{K} E_{k,p}\theta^{m_{k,p}}u \le \theta u, \qquad p \in N_0. \tag{4.14}$$

Now, let $\delta > 0$ be such that $|z^0 - z^*| \le \delta u$. Then by making use of (4.11) and (4.14) we have

$$|z^{p+1} - z^*| \le H_p H_{p-1} \cdots H_0 \delta u \le \theta^{p+1}\delta u \longrightarrow 0 \qquad (p \to \infty).$$

That is to say, $\lim\limits_{p\to\infty} z^p = z^*$, and hence, the proof of this theorem is fulfilled.

Compared with Theorem 4.1, no restriction is imposed on the weighting matrices in Theorem 4.2. Hence, we can choose them suitably such that Method 3.1 achieves high parallel efficiency. On the other hand, the restriction on the matrix splittings in Theorem 4.2 is quite different from that in Theorem 4.1. In fact, both of these two classes of restrictions on the matrix splittings are standard conditions for guaranteeing the convergence of iterative methods for solving the LCP(M,q), which naturally originate from the classical conditions for assuring the convergence of iterative methods for solving the linear systems of equations.

5. Parallel Chaotic Multisplitting Relaxation Methods

In Method 3.1, at every iterate each processor needs to solve implicitly a linear complementarity problem of smaller size. This makes it less convenient in concrete applications. In this section, we will present several explicit forms of Method 3.1, which are convenient for practical implementations.

For $M = (m_{ij}) \in \mathbb{R}^{n\times n}$, denote $D = diag(M) = diag(m_{11}, m_{22}, \cdots, m_{nn})$. For $k = 1, 2, \cdots, K$ and $p \in N_0$, let $L_{k,p} = (l_{ij}^{(k,p)}) \in \mathbb{R}^{n\times n}$ be strictly lower triangular matrices, $W_{k,p} = (w_{ij}^{(k,p)}) \in \mathbb{R}^{n\times n}$ be zero-diagonal matrices, and $E_{k,p} = diag(e_i^{(k,p)}) \in \mathbb{R}^{n\times n}$ be nonnegative diagonal matrices, such that

(1) $M = D + L_{k,p} + W_{k,p}$, $k = 1, 2, \cdots, K$, $p \in N_0$;
(2) $\det(D) \ne 0$;
(3) $\sum\limits_{k=1}^{K} E_{k,p} = I$, $p \in N_0$.

Given real constants γ_k and $\omega_k(\neq 0)$, $k=1,2,\cdots,K$. Take

$$\begin{cases} B_{k,p}=\dfrac{1}{\omega_k}(D+\gamma_k L_{k,p}), \\ C_{k,p}=\dfrac{1}{\omega_k}((1-\omega_k)D-(\omega_k-\gamma_k)L_{k,p}-\omega_k W_{k,p}), \end{cases} \qquad k=1,2,\cdots,K, \qquad p\in N_0$$

in Method 3.1. Then we obtain the following dynamic parallel chaotic multisplitting accelerated overrelaxation (AOR) method for solving the LCP(M,q).

Method 5.1. (Parallel Chaotic Multisplitting AOR Method for the LCP(M,q))

Step 1. Choose an arbitrary starting vector $z^0\in\mathbb{R}^n$. Set $p:=0$.

Step 2. For each $k\in\{1,2,\cdots,K\}$, set $z^{k,p,0}:=z^p$, and take a positive integer $m_{k,p}$.

Step 3. For each $k\in\{1,2,\cdots,K\}$ and $m=1$ to $m_{k,p}$, let $z^{k,p,m}$ be calculated according to

$$[z^{k,p,m}]_i=\begin{cases} 0, \quad \text{if} \quad \gamma_k\sum\limits_{j=1}^{i-1} l_{ij}^{(k,p)}([z^{k,p,m}]_j-[z^{k,p,m-1}]_j) \\ \qquad\qquad +\omega_k[Mz^{k,p,m-1}+q]_i>m_{ii}[z^{k,p,m-1}]_i, \\ [z^{k,p,m-1}]_i+\frac{\gamma_k}{m_{ii}}\sum\limits_{j=1}^{i-1} l_{ij}^{(k,p)}([z^{k,p,m-1}]_j-[z^{k,p,m}]_j)-\frac{\omega_k}{m_{ii}}[Mz^{k,p,m-1}+q]_i, \\ \qquad\qquad \text{otherwise}. \end{cases}$$

Step 4. For each $k\in\{1,2,\cdots,K\}$, set $z^{k,p+1}:=z^{k,p,m_{k,p}}$.

Step 5. $z^{p+1}=\sum\limits_{k=1}^{K}E_{k,p}z^{k,p+1}$.

Step 6. If $z^{p+1}=z^p$, then stop. Otherwise, set $p:=p+1$ and return to Step 2.

If we choose the parameter pairs (γ_k,ω_k), $k=1,2,\cdots,K$, to be (ω_k,ω_k), $k=1,2,\cdots,K$, $(1,1)$ and $(0,1)$, respectively, Method 5.1 gives the dynamic chaotic multisplitting SOR, Gauss-Seidel and Jacobi methods, correspondingly, for the LCP(M,q). Hence, an extensive sequence of parallel chaotic multisplitting relaxation methods can be obtained, which is quite practical and efficient for solving the large sparse linear complementarity problems on the high-speed multiprocessor systems. Moreover, through suitable adjustments of the parameters (γ_k,ω_k), $k=1,2,\cdots,K$, the convergence property of the chaotic multisplitting AOR method can be substantially improved. We remark that Method 5.1 reduces to the multisplitting AOR method for the LCP(M,q) in [1] when the composition numbers satisfying $m_{k,p}=1$ $(k=1,2,\cdots,K,p\in N_0)$.

We have the following convergence theorems for Method 5.1.

Theorem 5.1. *Let $M\in\mathbb{R}^{n\times n}$ be a symmetric positive definite matrix, and for every $p\in N_0$,*

$$f(\sum_{k=1}^{K}E_{k,p}z^{k,p})\le\max_{1\le k\le K}f(z^{k,p}).$$

Let $m_{k,p}(k=1,2,\cdots,K,p\in N_0)$ be positive integers bounded uniformly from above. Then the sequence $\{z^p\}_{p\in N_0}$ generated by Method 5.1 converges, independently of the positive integer sequences $\{m_{k,p}\}_{p\in N_0}$ $(k=1,2,\cdots,K)$, to the unique solution of the LCP(M,q), provided the relaxation parameters γ_k and ω_k, $k=1,2,\cdots,K$, satisfy $\gamma_k\in\mathbb{R}^1$ and $\omega_k>0(k=1,2,\cdots,K)$.

Proof. It is direct through verifying the assumptions of Theorem 4.1 one by one. Hence, it is omitted.

Theorem 5.2. *Let $M\in\mathbb{R}^{n\times n}$ be an H_+-matrix, $D=diag(M)$ and $B=M-D$. Assume that for every $p\in N_0$,*

$$|L_{k,p}|+|W_{k,p}|=|B|, \qquad k=1,2,\cdots,K, \quad p\in N_0.$$

Then the sequence $\{z^p\}_{p\in N_0}$ generated by Method 5.1 converges, independently of the positive integer sequences $\{m_{k,p}\}_{p\in N_0}$ $(k=1,2,\cdots,K)$, to the unique solution of the LCP(M,q), provided the relaxation parameters γ_k and ω_k, $k=1,2,\cdots,K$, satisfy

$$0\le\gamma_k\le\omega_k,\qquad 0<\omega_k<\frac{2}{1+\rho(D^{-1}|B|)},\qquad k=1,2,\cdots,K.$$

Proof. Because, for each $k\in\{1,2,\cdots,K\}$ and each $p\in N_0$,

$$\begin{cases}\langle B_{k,p}\rangle=\dfrac{1}{\omega_k}(D-\gamma_k|L_{k,p}|):=\widehat{\mathcal{B}}_{k,p},\\ |C_{k,p}|\le\dfrac{1}{\omega_k}[|1-\omega_k|D+(\omega_k-\gamma_k)|L_{k,p}|+\omega_k|W_{k,p}|]:=\widehat{\mathcal{C}}_{k,p},\end{cases}$$

$\widehat{\mathcal{B}}_{k,p}$ is an M-matrix and $\widehat{\mathcal{C}}_{k,p}$ a nonnegative matrix, and

$$\begin{aligned}\widehat{\mathcal{B}}_{k,p}-\widehat{\mathcal{C}}_{k,p}&=\frac{1}{\omega_k}[(1-|1-\omega_k|)D-\omega_k(|L_{k,p}|+|W_{k,p}|)]\\&=\frac{1}{\omega_k}[(1-|1-\omega_k|)D-\omega_k|B|]:=\widehat{M}_k,\end{aligned}$$

we see that for each $k\in\{1,2,\cdots,K\}$, $(\widehat{\mathcal{B}}_{k,p},\widehat{\mathcal{C}}_{k,p})$ $(p\in N_0)$ are H_+-compatible splittings of the matrix $\widehat{M}_k$. Considering $\rho(D^{-1}|B|)<1$ as $M\in\mathbb{R}^{n\times n}$ is an H_+-matrix, we can easily verify that $\widehat{M}_k$ $(k=1,2,\cdots,K)$ are H_+-matrices, too. Now, analogous to the proof of Theorem 4.2, we can demonstrate that $\{z^p\}_{p\in N_0}$ converges to z^*, the unique solution of the LCP(M,q).

6. Numerical Results

We consider the linear complementarity problem with the system matrix and given vector

$$M=\begin{pmatrix}R & -I & & & \\ -I & R & -I & & \\ & \ddots & \ddots & \ddots & \\ & & -I & R & -I\\ & & & -I & R\end{pmatrix}\in\mathbb{R}^{n\times n},\qquad q=\begin{pmatrix}10\\-10\\\vdots\\(-10)^{n-1}\\(-10)^n\end{pmatrix}\in\mathbb{R}^n,$$

respectively, where $R=tridiag(-1,4,-1)\in\mathbb{R}^{\tilde{n}\times\tilde{n}}$, $I\in\mathbb{R}^{\tilde{n}\times\tilde{n}}$ is the identity matrix, and $n=\tilde{n}^2$.

The test methods used in our numerical experiments are: (a) the sequential successive overrelaxation method (SOR) and the sequential accelerated overrelaxation method (AOR) in [6]; (b) the multisplitting successive overrelaxation method (MSOR) and the multisplitting accelerated overrelaxation method (MAOR) in [14,1]; and the chaotic multisplitting successive overrelaxation method (CMSOR) and the chaotic multisplitting accelerated overrelaxation method (CMAOR) in this paper.

The parallel machine used in our computations is an SGI Power Challenge multiprocessor computer. It consists of four 75 MHz TFP 64-bit RISC processors. These CMOS processors each delivers a peak theorectical performance of 0.3 GFLOPS. The data cache size is 16 Kbytes.

In our computations, we take

$$J_k=\{\tilde{n}_{k-1}\tilde{n}+1,\tilde{n}_{k-1}\tilde{n}+2,\cdots,\tilde{n}_{k+1}\tilde{n}\},\qquad k=1,2,\cdots,K$$

with $\tilde{n}_k=Int\left(\frac{k\tilde{n}}{K+1}\right)$, $k=1,2,\cdots,K$, and

$$L_{k,p} = (\mathcal{L}_{ij}^{(k,p)}) \in \mathbb{R}^{n\times n}, \quad \mathcal{L}_{ij}^{(k,p)} = \begin{cases} m_{ij}, & \text{for } i,j \in J_k \text{ and } i > j, \\ 0, & \text{otherwise,} \end{cases} \qquad i,j = 1,2,\cdots,n,$$

$$W_{k,p} = (\mathcal{W}_{ij}^{(k,p)}) \in \mathbb{R}^{n\times n}, \quad \mathcal{W}_{ij}^{(k,p)} = \begin{cases} 0, & \text{for } i = j, \\ 0, & \text{for } i,j \in J_k, \qquad i,j = 1,2,\cdots,n, \\ m_{ij}, & \text{otherwise,} \end{cases}$$

$$E_{k,p} = diag(e_i^{(k,p)}) \in \mathbb{R}^{n\times n}, \quad e_i^{(k,p)} = \begin{cases} 1, & \text{for } 1 \le i \le \tilde{n}_1\tilde{n}, \quad k = 1, \\ 0.5, & \text{for } \tilde{n}_{k-1}\tilde{n} + 1 \le i \le \tilde{n}_k\tilde{n}, \quad 2 \le k \le K, \\ 0.5, & \text{for } \tilde{n}_k\tilde{n} + 1 \le i \le \tilde{n}_{k+1}\tilde{n}, \quad 1 \le k \le K-1, \\ 1, & \text{for } \tilde{n}_K\tilde{n} + 1 \le i \le n, \quad k = K. \end{cases}$$

The LCP is solved by the aforementioned test methods corresponding to various problem sizes and relaxation parameters when the processor number K is respectively taken to be $K = 2$, $K = 3$ and $K = 4$. All our computations are started from an initial vector having all components equal to 40.0, and terminated once the current iterations z^p obey

$$\frac{|(z^p)^T(Mz^p + q)|}{|(z^0)^T(Mz^0 + q)|} \le 10^{-6}.$$

From the numerical computations we see that in the sense of CPU time and the parallel efficiency, the chaotic multisplitting relaxation methods are superior to the corresponding multisplitting relaxation methods, and the AOR-like multisplitting methods are superior to the corresponding SOR-like multisplitting methods. In particular, the advantages of the CMAOR method over the CMSOR method are that: (i) when the latter diverges, the former can still converge; (ii) when the latter converges, the former converges faster with higher parallel efficiency; and (iii) the former is less sensitive to the relaxation parameters and it has larger convergence domain than the latter. Therefore, we can conclude that the new parallel chaotic multisplitting relaxation methods have better numerical properties than their corresponding synchronous parallel alternatives.

For $n = 6400$ and $K = 3$, some of the numerical results are listed in Tables (I)-(II). Here, we use CPU to denote the CPU time required for an iteration to attain the above stopping criterions, ∞ to denote that an iteration does not satisfy the stopping criterions after 5000 iterations, and SP to denote the speed-up of a multisplitting relaxation method, which is defined to be the ratio of the CPU times of the sequential relaxation method to the corresponding multisplitting relaxation method. The numerical results for two and four processor cases are much similar to the three processor case.

Table (I) CPUs and SPs for the SOR-like methods

ω		1	1.1	1.2	1.4	1.6	1.8	1.9	1.95	1.97	1.99
SOR	CPU	50.50	41.35	33.67	21.63	12.61	5.56	2.53	1.68	3.32	4.26
MSOR	CPU	27.37	22.44	18.30	11.85	7.03	3.25	1.66	1.14	1.96	4.31
	SP	1.85	1.84	1.84	1.82	1.79	1.71	1.52	1.47	1.69	0.98
CMSOR	CPU	24.58	19.96	16.41	10.77	6.37	2.92	1.58	0.80	0.51	0.85
	SP	2.05	2.07	2.05	2.00	1.98	1.90	1.60	2.1	6.51	5.01

Table (II) CPUs and SPs for the AOR-like methods

γ		1	1.1	1.2	1.4	1.6	1.8	1.9	1.95	1.97	1.99
ω		1.2	1.3	1.3	1.5	1.7	1.9	1.94	1.85	1.98	1.8
AOR	CPU	42.11	35.08	31.11	20.20	11.87	5.29	2.26	1.46	3.47	1.80
MAOR	CPU	22.86	19.01	16.89	11.11	6.61	3.07	1.38	0.94	4.22	1.10
	SP	1.84	1.85	1.84	1.82	1.80	1.72	1.64	1.59	0.82	1.64
CMAOR	CPU	20.32	16.99	15.18	10.12	5.98	2.83	1.50	0.93	0.59	0.80
	SP	2.07	2.06	2.05	2.00	1.98	1.89	1.51	1.57	5.88	2.25

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