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AN EXTREMAL APPROACH TO BIRKHOFF QUADRATURE FORMULAS*1)

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Abstract

As we know, a solution of an extremal problem with Hermite interpolation constraints is a system of nodes of corresponding Gaussian Hermite quadrature formula (the so-called Jacobi approach). But this conclusion is violated for a Birkhoff quadrature formula. In this paper an extremal problem with Birkhoff interpolation constraints is discussed, from which a large class of Birkhoff quadrature formulas may be derived.

Key words: An extremal approach, Birkhoff quadrature formulas.

1. Introduction and Main Results

In this paper we shall use the definitions and notations of [3]. Let $E = (e_{ik})_{i=0,\ k=0}^{m+1\ n}$ be an incidence matrix with entries consisting of zeros and ones and satisfying $|E| := \sum_{i,k} e_{ik} = n+1$ (here we allow a zero row). Furthermore, in what follows we assume that

(A) E satisfies the Pólya condition

$$\sum_{i=0}^{m+1}\sum_{k=0}^{r} e_{ik} \geq r+1, \quad r = 0, 1, ..., n; \tag{1.1}$$

(B) all sequences of E in the interior rows, $0 < i < m+1$, are even.

Let S_m denote the set of points $X = (x_0, x_1, ..., x_m, x_{m+1})$ for which

$$0 = x_0 < x_1 < ... < x_m < x_{m+1} = 1 \tag{1.2}$$

and $\overline{S}_m$ its clousure. If some of the coordinates of $X \in \overline{S}_m$ coincide, E is replaced by its corresponding coalescence [3, p. 27]. Then by the Atkinson-Sharma Theorem [3, p. 10] the pair (E, X) is regular for all $X \in \overline{S}_m$ and the quadrature formula of the form

$$\int_0^1 f(x)dg(x) = \sum_{e_{ik}=1} a_{ik} f^{(k)}(x_i) \tag{1.3}$$

is exact for all $f \in \mathbf{P}_n$, the space of all polynomials of degree at most n, where $g(x)$ is a strictly increasing function.

Among all quadrature formulas particularly interesting is the one which is derived from the extremal problem:

$$\int_0^1 |\Omega(E, X; x)| dg(x) = \min_{Y \in \overline{S}_m} \int_0^1 |\Omega(E, Y; x)| dg(x), \tag{1.4}$$

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where $\Omega(x) := \Omega(E, X; x) = x^{n+1} + ...$ satisfies

$$\Omega^{(k)}(x_i) = 0, \quad e_{ik} = 1 \quad e_{ik} \in E. \tag{1.5}$$

As pointed out in [1], the quantity of the left side in (1.4) is the major term in the estimate of the error of (1.3). Meanwhile, as we know, a sulution of the extremal problem (1.4) with Hermite interpolation constraints must be the system of nodes of corresponding Gaussian Hermite quadrature formula (1.3) (the so-called Jacobi approach). But this conclusion is not valid for a Birkhoff quadrature formula, the reason is that the basic condition (1.2) may be violated. An important question is whether a solution X of (1.4) satisfies (1.2)? Only few papers discuss this question for a proper Birkhoff quadratrue formula. One of them is given by K. Jetter [2]. The main aim of this paper will give a sufficient condition that a solution X of (1.4) satisfy (1.2), from which a class of Birkhoff quadrature formulas may be derived. To state our results, for each i, $0 \le i \le m+1$, let k_i denote the smallest index k such that $e_{ik} = 1$ (when the i-th row is a zero row, we assume $k_i = +\infty$). Put

$$\mu_i = \min\{k_i, k_{i+1}\}, \quad n_i = \sum_{k=0}^{n} (e_{ik} + e_{i+1,k}), \quad i = 0, 1, ..., m. \tag{1.6}$$

The main result in this paper is the following

Theorem. *Let an incidence matrix E satisfy the conditions* (A) *and* (B). *Assume that*

(C) *there is an index I, $0 \le I \le m$, such that*

$$\begin{cases} \mu_{i+1} \le \mu_i, & i < I, \\ \mu_i \le \mu_{i+1}, & i \ge I; \end{cases} \tag{1.7}$$

(D) *for each i, $1 \le i \le m-1$,*

$$\sum_{k=\mu_i}^{\mu_i + r} (e_{ik} + e_{i+1,k}) \ge r + 1, \quad r = 0, 1, ..., n_i - 1 \tag{1.8}$$

and

$$e_{i,\mu_i + n_i - 1} = e_{i+1,\mu_i + n_i - 1} = 0. \tag{1.9}$$

Then each solution of (1.4) satisfies (1.2).

Moreover, (1.3) is exact for all $f \in \mathbf{P_n}$, where

$$\sum_{\substack{k=0 \\ e_{ik}=1}}^{n} a_{ik} \Omega^{(k+1)}(E, X; x_i) = 0, \quad i = 1, ..., m. \tag{1.10}$$

A specail case of this theorem when each interior row of E contains only one sequence can be found in [2, Theorem 5.1].

In the next section we derive some auxiliary lemmas. The proof of the theorem is put in Section 3. In the last section we give a remark. Our proofs use many ideas of [1,2].

2. Auxiliary Lemmas

First we derive some properties of the polynomials $\Omega(x)$.

Lemma 1. [2, Lemma 2.2] *Let E satisfy the conditions (A) and (B).*

(a) *The polynomials $\Omega(x)$ depend continuously on $X \in \overline{S}_m$.*

(b) *For all $X \in \overline{S}_m$ we have $(-1)^\epsilon \Omega(x) \ge 0$, $x \in [0, 1]$, where ϵ is the number of entries $e_{ik} = 1$ in the last row of E.*

(c) *For* $X \in S_m$ *and* $k = 0, ..., n$, *we have* $\Omega^{(k)}(0) = 0$ *if and only if* $e_{0,k} = 1$, *or equivalently,* $\Omega^{(k)}(\delta)\Omega^{(k+1)}(\delta) > 0$ *for small* $\delta > 0$. *Similarly,* $\Omega^{(k)}(1) = 0$ *if and only if* $e_{m+1,k} = 1$, *or equivalently,* $\Omega^{(k)}(1-\delta)\Omega^{(k+1)}(1-\delta) < 0$ *for small* $\delta > 0$.

(d) *Property (c) holds true for any* $X \in \overline{S}_m$ *if the members* $e_{0,k}$ *and* $e_{m+1,k}$ *are replaced by the entries in exterior rows of the corresponding coalesced matrix* E'.

As an immediate consequence of Lemma 1 we state

Lemma 2. *Let* E *satisfy the conditions (A) and (B). Then*

$$\begin{cases} sgn\ \Omega^{(j)}(0) = (-1)^{\epsilon + j + \Sigma_{k<j} e_{0k}}, & e_{0j} = 0, \\ sgn\ \Omega^{(j)}(1) = (-1)^{\epsilon + \Sigma_{k<j} e_{m+1,k}}, & e_{m+1,j} = 0. \end{cases} \tag{2.1}$$

Lemma 3. *Let* E *satisfy the conditions (A) and (B). Then*

$$\|\Omega^{(k)}\| \le (n+1)!, \quad k = 0, 1, ..., n+1. \tag{2.2}$$

Here $\|\cdot\|$ *stands for the Chebyshev norm.*

Proof. The conditions (A) and (B) imply that for each k, $0 \le k \le n$, there is a $\xi_k \in [0,1]$ such that

$$\Omega^{(k)}(\xi_k) = 0, \quad k = 0, 1, ..., n. \tag{2.3}$$

Thus

$$|\Omega^{(k-1)}(x)| = \left| \int_{\xi_{k-1}}^{x} \Omega^{(k)}(t)dt \right| \le \|\Omega^{(k)}\|, \quad k = 1, ..., n+1,$$

which, coupled with $\Omega^{(n+1)}(x) = (n+1)!$, yields (2.2).

Lemma 4. *If* $f \in \mathbf{P_n}$ *has two adjacent zeros* $y_1 < y_2$ *and* f' *has a unique zero* y *in* (y_1, y_2) *then*

$$\min\{|y - y_1|, |y - y_2|\} \ge \frac{1}{2n^2}(y_2 - y_1).$$

Proof. This follows from Markoff's Inequakity. The details may be found, say, in [2, p. 1087].

Next, we give the main result in this section as follows.

Lemma 5. *Let an incidence matrix* E *satisfy the conditions (A)-(D) and let* E' *be obtained from* E *by coalescing the* $(j+1)$*-th row to the* j*-th row for a fixed* j, $0 \le j \le m$. *Assume that for* $X' = (x_0, ..., x_{j-1}, \xi, x_{j+2}, ..., x_{m+1})$ *with*

$$0 = x_0 < ... < x_{j-1} < \xi < x_{j+2} < ... < x_{m+1} = 1$$

we have

$$\int_0^1 |\Omega(E', X'; x)| dg(x) = \min_{Y \in \overline{S}_{m-1}} \int_0^1 |\Omega(E', Y; x)| dg(x). \tag{2.4}$$

Then there exists a point $X \in S_m$ *such that*

$$\int_0^1 |\Omega(E, X; x)| dg(x) < \int_0^1 |\Omega(E', X'; x)| dg(x). \tag{2.5}$$

Proof. For simplicity we write $\Omega_0(x) := \Omega(E', X'; x)$ and denote by $Z(f, [a,b])$ the number of zeros of f on $[a,b]$ counting multiplicities.

We distinguish two cases.

Case 1. $0 < j < m$.

Let x_j and x_{j+1} satisfy $x_{j-1} < x_j < \xi < x_{j+1} < x_{j+2}$. We are going to choose x_j and x_{j+1} so that (2.5) holds.

Claim 1. There is a $\delta_1 > 0$ such that for each $k,\ 0 \le k \le n_j$, we have

$$Z(\Omega^{(\mu_j+k)}, \Delta) = m_k - k, \quad 0 \le k \le n_j \tag{2.6}$$

for all $\delta \le \delta_1$, where

$$\Delta := [x_j, x_{j+1}], \quad \delta := x_{j+1} - x_j, \quad m_k := \sum_{r=\mu_j}^{\mu_j+k} (e_{jr} + e_{j+1,r}).$$

In particular, we have

$$Z(\Omega^{(\mu_j+n_j-k)}, \Delta) = k, \quad k = 0, 1, 2. \tag{2.7}$$

In fact, by (1.8) we have $Z(\Omega^{(\mu_j+k)}, \Delta) \ge m_k - k$ $(0 \le k \le n_j - 1)$. This inequality is also true for $k = n_j$, because $m_{n_j} - n_j = 0$. So to prove Claim 1 it suffices to show that

$$Z(\Omega^{(\mu_j+k)}, \Delta) \le m_k - k, \quad 0 \le k \le n_j$$

holds for all $\delta \le \delta_1$, where $\delta_1 > 0$ is a certain constant. Suppose to the contrary that for some $k,\ 0 \le k \le n_j$, we have $Z(\Omega^{(\mu_j+k)}, \Delta) \ge m_k - k + 1$ for any $\delta > 0$. If $k = n_j$ then this means $Z(\Omega^{(\mu_j+n_j)}, \Delta) \ge 1$. If $k < n_j$ then this stipulates at least $m_k - k$ Rolle's zeros of $\Omega^{(\mu_j+k+1)}(x)$ in (x_j, x_{j+1}). Thus

$$Z(\Omega^{(\mu_j+k+1)}, \Delta) \ge m_k - k + e_{j,\mu_j+k+1} + e_{j+1,\mu_j+k+1} = m_{k+1} - k = m_{k+1} - (k+1) + 1.$$

That is to say, by induction we can again obtain that $Z(\Omega^{(\mu_j+n_j)}, \Delta) \ge 1$, or equivalently, there is a $\xi_\delta \in \Delta$ such that $\Omega^{(\mu_j+n_j)}(\xi_\delta) = 0$. Since $\delta > 0$ is arbitrary, we get $\Omega_0^{(\mu_j+n_j)}(\xi) = 0$ as $\delta \to 0$ by Lemma 1. On the other hand, Condition (C) implies that the sequence of 1's in the j-th row of E' is not supported and hence $\Omega_0^{(\mu_j+n_j)}(\xi) = 0$ would lead to $\Omega_0 \equiv 0$, a contradiction.

(2.7) follows directly from (2.6), (1.8), and (1.9).

This proves Claim 1.

For an arbitrary but fixed $\delta,\ 0 < \delta < \min\{\delta_1, \xi - x_{j-1}, x_{j+2} - \xi\}$, when x_j moves from $\xi - \delta$ to ξ by Lemmas 1 and 4 the unique zero of $\Omega^{(\mu_j+n_j-1)}(x)$ in Δ must move from the left side of ξ to the right. So we can choose x_j and $x_{j+1},\ x_{j-1} < x_j < \xi < x_{j+1} < x_{j+2}$, such that

$$\Omega^{(\mu_j+n_j-1)}(\xi) = 0. \tag{2.8}$$

By Claim 1 we have that for small $\delta > 0$

$$\Omega^{(\mu_j+n_j-2)}(\xi)\Omega^{(\mu_j+n_j)}(\xi) < 0$$

and hence

$$\Omega^{(\mu_j+n_j-2)}(\xi)\Omega_0^{(\mu_j+n_j)}(\xi) < 0. \tag{2.9}$$

Applying the quadrature formula for the pair (E', X') to the polynomial $\Omega - \Omega_0$ yields

$$\int_0^1 |\Omega(x)| dg(x) - \int_0^1 |\Omega_0(x)| dg(x) = \int_0^1 (-1)^\epsilon [\Omega(x) - \Omega_0(x)] dg(x) = \sum_{k=\mu_j}^{\mu_j+n_j-2} \lambda_k \Omega^{(k)}(\xi). \tag{2.10}$$

Since the sequence of 1's in the j-th row of E' is not supported, by the same arguments as in [2, p. 1088] we conclude

$$\operatorname{sgn}\, \Omega_0^{(\mu_j+n_j)}(\xi) = \operatorname{sgn}\, \lambda_{\mu_j+n_j-2}(\xi) = \epsilon_j, \tag{2.11}$$

$$\epsilon_j = \begin{cases} (-1)^{\mu_j}, & j < I, \\ 1, & j \geq I. \end{cases} \tag{2.12}$$

According to Claim 1 let $\xi_{1k} < \xi_{2k} < ... < \xi_{i_k,k}$ $(i_k := m_k - k)$ be the zeros of $\Omega^{(\mu_j+k)}$ on $\Delta,\ 0 \leq k \leq n_j - 1$.

Claim 2. If $i_{k-1} \geq 2$ and $i_k \geq 2$ for $k \geq 1$ then

$$\xi_{i_k,k} - \xi_{1k} \geq \frac{1}{2(n+1)^2}[\xi_{i_{k-1},k-1} - \xi_{1,k-1}]. \tag{2.13}$$

In fact, from $\xi_{1,k-1}, ..., \xi_{i_{k-1},k-1}$ we get $i_{k-1} - 1$ Rolle's zeros $\xi'_{1,k}, ..., \xi'_{i_{k-1}-1,k}$, which satisfy

$$\xi_{1,k-1} < \xi'_{1k} < \xi_{2,k-1} < ... < \xi_{i_{k-1}-1,k-1} < \xi'_{i_{k-1}-1,k} < \xi_{i_{k-1},k-1}.$$

Of course $\{\xi'_{1k}, ..., \xi'_{i_{k-1}-1,k}\} \subset \{\xi_{1k}, ..., \xi_{i_k,k}\}$.

By means of Lemma 4 we obtain

$$|\xi'_{1k} - \xi_{2,k-1}| \geq \frac{1}{2(n+1)^2}|\xi_{1,k-1} - \xi_{2,k-1}|,$$

$$|\xi_{i_{k-1}-1,k-1} - \xi'_{i_{k-1}-1,k}| \geq \frac{1}{2(n+1)^2}|\xi_{i_{k-1}-1,k-1} - \xi_{i_{k-1},k-1}|.$$

Adding these two inequalities and then adding $\xi_{i_{k-1}-1,k-1} - \xi_{2,k-1}$ to the both sides of the resulting inequality yields

$$|\xi'_{i_{k-1}-1,k} - \xi'_{1k}| \geq \frac{1}{2(n+1)^2}|\xi_{i_{k-1},k-1} - \xi_{1,k-1}|.$$

Since $\xi_{i_k,k} - \xi_{1k} > \xi'_{i_{k-1}-1,k} - \xi'_{1k}$, the claim follows.

As an immediate consequence of Claim 1 we state the following

Claim 3. If r is the largest index $k \leq n_j - 2$ such that $i_k = 1$, then $e_{j,\mu_j+r+1} = e_{j+1,\mu_j+r+1} = 1$ and $i_k \geq 2,\ r+1 \leq k \leq n_j - 2$.

Combining Claims 2 and 3 directly yields

Claim 4. We have

$$\xi_{2,n_j-2} - \xi_{1,n_j-2} \geq \frac{\delta}{[2(n+1)^2]^n}. \tag{2.14}$$

Claim 5. There exists a $\delta_2 > 0$ such that for all $\delta \leq \delta_2$ we have

$$|\Omega^{(\mu_j+k)}(\xi)| \leq c_1 |\Omega^{(\mu_j+n_j-2)}(\xi)| \delta^{n_j-k-2}, \quad k = 0, 1, ..., n_j - 3. \tag{2.15}$$

Here c_1 is a constant depending only on n.

Indeed, using $\Omega^{(\mu_j+k)}(\xi_{1k}) = 0, \quad 0 \leq k \leq n_j - 1$, and the Taylor expression for $\Omega(x)$

$$\Omega(x) = \sum_{r=0}^{n+1} \frac{1}{r!} \Omega^{(r)}(\xi)(x-\xi)^r, \tag{2.16}$$

we obtain

$$\Omega^{(\mu_j+k)}(\xi) = -\sum_{r=\mu_j+k+1}^{n+1} \frac{1}{(r-\mu_j-k)!} \Omega^{(r)}(\xi)(\xi_{1k}-\xi)^{r-\mu_j-k}, \quad k = 0, 1, ...n_j - 1. \tag{2.17}$$

Hence

$$|\Omega^{(\mu_j+k)}(\xi)| \leq \sum_{r=\mu_j+k+1}^{n+1} |\Omega^{(r)}(\xi)| \delta^{r-\mu_j-k}, \quad k = 0, 1, ...n_j - 1. \tag{2.18}$$

It follows from (2.2), (2.8), and (2.18) by induction that

$$|\Omega^{(\mu_j+k)}(\xi)| \le c_2\delta^{n_j-k}, \quad k=0,1,...n_j-1. \tag{2.19}$$

On the other hand, by Lemma 4 it follows from (2.17) and (2.14) that for small $\delta > 0$

$$\begin{aligned}
&|\Omega^{(\mu_j+n_j-2)}(\xi)|\\
\ge&\frac{1}{2}|\Omega^{(\mu_j+n_j)}(\xi)|(\xi_{1,n_j-2}-\xi)^2 - \left|\sum_{r=\mu_j+n_j+1}^{n+1}\frac{\Omega^{(r)}(\xi)}{(r-\mu_j-n_j+2)!}(\xi_{1,n_j-2}-\xi)^{r-\mu_j-n_j+2}\right|\\
\ge&\frac{1}{4}|\Omega^{(\mu_j+n_j)}(\xi)|(\xi_{1,n_j-2}-\xi)^2\\
\ge&\frac{1}{4[2(n+1)^2]^2}|\Omega^{(\mu_j+n_j)}(\xi)|(\xi_{1,n_j-2}-\xi_{2,n_j-2})^2\\
\ge&\frac{1}{4[2(n+1)^2]^{2n+2}}|\Omega^{(\mu_j+n_j)}(\xi)|\delta^2,
\end{aligned}$$

which, coupled with (2.19), gives (2.15).

Now in this case the lemma follows from (2.9)-(2.11) and (2.15).

Case 2. $j=0$ or $j=m$.

We give the proof for the case $j=0$ only, the one for the case $j=m$ being similar.

In this case $X'=(x_0,x_2,...,x_{m+1})\in S_{m-1}$. We apply the quadrature formula for the pair (E',X') to $\Omega-\Omega_0$ to get

$$\int_0^1|\Omega(x)|dg(x)-\int_0^1|\Omega_0(x)|dg(x)=\sum_{\substack{k=\mu_0\\ e'_{0k}=1}}^{n+1}\lambda_k\Omega^{(k)}(0). \tag{2.20}$$

Let $e'_{0p}=...=e'_{0r}=1$ be an arbitrary sequence of 1's in the 0-th row of E', i.e., $p=0$ or $e'_{0,p-1}=0$ as well as $r=n$ or $e'_{0,r+1}=0$, which satisfies $\Omega^{(k)}(0)\ne 0$ for some $k,\ p\le k\le r$. Put $q=\max\{k:\ \Omega^{(k)}(0)\ne0,\ p\le k\le r\}$. Then it is easy to see that

$$\sum_{k\le q}e'_{0k}=\sum_{k\le q}(e_{0k}+e_{1k}).$$

Since $\sum_{k\le q}e_{1k}$ is even, $\sum_{k\le q}(e'_{0k}+e_{0k})$ is even and hence $\sum_{k<q}(e'_{0k}+e_{0k})$ is odd.

Let E^* be obtained from E' by replacing the 1 in position $(0,q)$ by 0 and let $\Omega_1(x):=\Omega(E^*,X';x)$. Then by Lemma 2

$$\text{sgn }[\Omega^{(q)}(0)\Omega_1^{(q)}(0)]=(-1)^{\Sigma_{k<q}(e'_{0k}+e_{0k})}=-1.$$

On the other hand, we see that the polynomial $\Omega_1(x)/\Omega_1^{(q)}(0)$ of degree n is the fundamental function for the pair (E^*,X') corresponding to position $(0,q)$ and does not change sign on $[0,1]$. Hence

$$\text{sgn }\lambda_q=(-1)^\epsilon\text{sgn }\left\{\frac{\Omega_1(\delta)}{\Omega_1^{(q)}(0)}\right\}=\text{sgn }\Omega_1^{(q)}(0).$$

Thus

$$\lambda_q\Omega^{(q)}(0)<0.$$

This shows that

$$\sum_{k=p}^{q}\lambda_k\Omega^{(k)}(0)<0$$

for small $\delta > 0$. Then the right term of (2.20) is negative for small $\delta > 0$, which proves (2.5).

3. Proof of Theorem

Suppose to contrary that (1.2) does not hold for a solution X of (1.4). If we discard multiplicities in the coordinates of X, we get the point

$$X^* = (x_0^*, x_1^*, ..., x_{\mu+1}^*), \quad 0 = x_0^* < x_1^* < ... < x_{\mu+1}^* = 1, \quad \mu < m, \tag{3.1}$$

in which, say, $x_j^* = x_p = x_{p+1} = ... = x_q, \quad 0 \le p < q \le m+1$. Let E^* be the corresponding coalescence of E and let $\overline{E}$ be obtained from E^* by decoalescing the j-th row into two rows: the first one corresponding to the p-th row of E and the other corresponding to coalescence of rows $i = p+1, ..., q$ of E. Clearly, we have

$$\int_0^1 |\Omega(E^*, X^*; x)| dg(x) = \min_{Y \in \overline{S}_\mu} \int_0^1 |\Omega(E^*, Y; x)| dg(x). \tag{3.2}$$

We note that any coalescence of E again satisfies the conditions (A)-(D). According to Lemma 5 we conclude that there must exist a point

$$\overline{X} = \{x_0^*, ..., x_{j-1}^*, x', x'', x_{j+1}^*, ..., x_{\mu+1}^*\} \in S_{\mu+1}$$

such that

$$\int_0^1 |\Omega(\overline{E}, \overline{X}; x)| dg(x) < \int_0^1 |\Omega(E^*, X^*; x)| dg(x), \tag{3.3}$$

which contradicts the assumption. This proves the first conclusion of the theorem.

As an immediate consequence of the above proved result, using necessary conditions for a solution of (1.4) satisfying (1.2) which are derived by Jetter in [2] with the method of Lagrangian multipliers, we can conclude that (1.3) is exact for all $f \in \mathbf{P_n}$ and (1.10) holds.

4. A Remark

The following example shows that the condition (1.9) may not be omitted in general, even if the two subsequent rows of E have a common column of entries $e_{ik} = 1$.

Example. Let $g(x) \equiv x$ and

$$E = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then $x_1 = x_2 = \frac{1}{2}$ is the unique solution of (1.4).

In fact, it is easy to check that ($y := x_2 - x_1$)

$$\Omega(x) = 3y^4(x - x_1)^2 - 4y^3(x - x_1)^3 + (x - x_1)^6,$$

$$\begin{aligned} \int_0^1 \Omega(x) dx =& y^4(3x_1^2 - 3x_1 + 1) + y^3(4x_1^3 - 6x_1^2 + 4x_1 - 1) \\ &+ x_1^6 - 3x_1^5 + 5x_1^4 - 5x_1^3 + 3x_1^2 - x_1 + \frac{2}{7} \\ :=& f(x_1, y). \end{aligned}$$

$\frac{\partial f}{\partial x_1} = 0$ gives

$$3y^4(2x_1 - 1) + 4y^3(3x_1^2 - 3x_1 + 1) + (2x_1 - 1)(3x_1^4 - 6x_1^3 + 7x_1^2 - 4x_1 + 1) = 0; \tag{4.1}$$

$\frac{\partial f}{\partial y} = 0$ gives

$$4y^3(3x_1^2 - 3x_1 + 1) + 3y^2(2x_1 - 1)(2x_1^2 - 2x_1 + 1) = 0. \tag{4.2}$$

(4.1) minus (4.2) yields

$$(2x_1 - 1)[12(y^2 - x_1^2 + x_1 - \frac{1}{2})^2 + (2x_1 - 1)^2] = 0.$$

This, coupled with (4.2), shows that $x_1 = \frac{1}{2}$, $y = 0$ is the unique solution of the system of equations.

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