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MULTIPLICATIVE SCHWARZ ALGORITHM WITH TIME STEPPING ALONG CHARACTERISTIC FOR CONVECTION DIFFUSION EQUATIONS*1)

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Abstract

Based on domain decomposition, we give two multiplicative schwarz methods with time stepping along characteristic for semi-linear, convection diffusion parabolic problems. we give some a priori error estimates, which tell us that the convergence of the approximate solution are independent of the iteration times at every time-level. Finally we give some numerical examples.

Key words: Multiplicative Schwarz method, Convection diffusion equation, Characteristic, Error estimate.

1. Introduction

Multiplicative Schwarz method, based on domain decomposition, is a powerful iteration methods for solving elliptic equations and other stationary problems. A systematic theory has been developed for elliptic finite element problems in the past few years, see [2, 5, 11, 12]. But there are little works of domain decomposition methods for time-dependence problems. In [11], Lions gives a kind of Schwarz alternating algorithm in two subdomain case for heat equations and gives a convergence result but does not give any error estimate. In [7, 8] Dawson and coworkers give a nonoverlapping domain decomposition method for parabolic equations, but since they use explicit schemes at intersection points, the stability condition $\Delta t \leq \frac{1}{2}H^2$ is needed. In [3, 4] Cai consider a kind of additive Schwarz algorithms and multiplicative Schwarz method and prove that the convergence rate is smaller than one for parabolic equations. In [13] the authors give the multiplicative Schwarz methods for linear parabolic problems.

In this paper we are interested in solving the convection diffusion problems using domain decomposition method. We use time-stepping along characteristic method mentioned by Douglas, Russell [9], which was powerful especially for convection-dominated equations, and Galerkin approximation in the space variables. At a fixed time level, the resulting equation is equivalent to an elliptic problem which depends on a time-step increment and the approximate solution at last time level. Therefore we can apply the multiplicative Schwarz method, originally proposed for elliptic equations to the convection diffusion equations at each time level. The crucial mathematical questions is then to know how the convergence and the error depend on the space mesh, the time step parameter and the number of iterations at each time level.

The outline of this paper is as follows. In Section 2 we present the continuous and discrete convection diffusion equations and give two kinds of multiplicative Schwarz algorithms with time-stepping along characteristic. We also give the optimal order error estimate results, which tell us that the approximate solutions converge after one cycle of iteration at each time level. In Section 3 we give some lemmas. In Section 4 we give proofs of theorems mentioned in Section 2. Finally in Section 5 we give some numerical examples.

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Throughout this paper, c or C ,with or without subscripts, denotes a generic, strictly positive constant. Its value may be different at different occurences, but is independent of the spacial meshsize h and time increment $\triangle t$, which will be introduced later.

2. Schwarz Algorithms and Convergence Results

Without loss of generality we consider the following model problem in a bounded polygonial domain $\Omega \subset R^2$

$$\frac{\partial u}{\partial t} + b \cdot \bigtriangledown u - \sum_{i,j=1}^{2} \frac{\partial}{\partial x_j}(a_{ij}\frac{\partial u}{\partial x_i}) = f(u) \text{ in } \Omega, \tag{1}$$

$$u = 0 \text{ on } \partial\Omega, \tag{2}$$

$$u(x,0) = u^0(x) \text{ in } \Omega, \tag{3}$$

where $b = (b_1, b_2)$, $b \cdot \bigtriangledown u = b_1 \dfrac{\partial u}{\partial x_1} + b_2 \dfrac{\partial u}{\partial x_2}$, J=(0,T] denote the time interval, $a_{ij} = a_{ji}$ and there exists a positive constant γ such that

$$\sum_{ij=1}^{2} a_{ij}\xi_i\xi_j \geq \gamma|\xi|^2 \quad \xi = (\xi_1, \xi_2)^\top \in R^2, \tag{4}$$

The standard variational formulation of the above problem is: Find $u(t) \in L^2(J; H_0^1(\Omega))$ such that

$$(\frac{\partial u}{\partial t}, v) + a(u,v) = (f(u), v) \quad v \in H_0^1(\Omega), \tag{5}$$

$$(u(0), v) = (u^0, v),$$

where

$$a(u,v) = \int_\Omega \sum_{ij=1}^{2} a_{ij} \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j} dx, \tag{6}$$

$$(f(u), v) = \int_\Omega f(u) v dx.$$

Let $\triangle t$ denote time increment, $t^n = n\triangle t, u^n = u(t^n)$, for any point $x = (x_1, x_2)$, let $\bar{x}$ denote the point along the approximate characteristic direction,

$$\bar{x} = \begin{cases} x - b\triangle t = (x_1 - b_1\triangle t, x_2 - b_2\triangle t) & when \quad x - b\triangle t \in \Omega, \\ 2Y(x) - X(x) & when \quad x - b\triangle t \notin \Omega, \end{cases}$$

where $Y(x) \in \partial\Omega$ denote the project point of $x, X(x) \in \Omega$ denote the symmetric point of x about $\partial\Omega$. We also let

$$\overline{u}^{n-1} = u(\overline{x}, t^{n-1}), \tag{7}$$

then[9,10,14]

$$\frac{u^n - \bar{u}^{n-1}}{\triangle t} = \frac{\partial u}{\partial t} + b. \bigtriangledown u + O(\frac{\partial^2 u}{\partial \tau^2}\tau), \tag{8}$$

where τ denote the unit vector at characteristic direction for the transfer term $(\dfrac{\partial u}{\partial t} + b. \bigtriangledown u)$. Then equation (1) can be approximated by

$$(\frac{u^n - \bar{u}^{n-1}}{\tau}, v) = (f(u^{n-1}), v) + (\rho^n, v), \tag{9}$$

where

$$\rho^n = \frac{u^n - \bar{u}^{n-1}}{\triangle t} - (\frac{\partial u}{\partial t} + b.\nabla u)^n + f(u^n) - f(u^{n-1}) = O(\frac{\partial^2 u}{\partial \tau^2}\triangle t + \triangle t).$$

Now we divide Ω into overlapping subdomains $\Omega_1, \Omega_2, ..., \Omega_p$ satisfying the following Condition (A):

Condition (A). For any $x \in \bar{\Omega}$ there exists a open domain D_x and $i \in \{1, 2, ..., p\}$ such that $x \in D_x$ and $D_x \cap \Omega \subset \Omega_i$.

Under condition (A) we can use the algorithm given below in practical implementation, but for convergence analyses we suppose some stronger conditions, the following Condition (B) or Condition (C), which are practical and reasonable, see [13] for detail.

Condition(B). The subregion $\Omega_i (1 \le i \le p)$ can be divided into four parts:

$$D_j = \sum_{r_{j-1}+1 \le i \le r_j} \Omega_i, \ j = 1, 2, 3, 4, \ r_0 = 0, \ r_4 = p. \tag{10}$$

Subdomains in D_j are disjoint and $\{D_1, D_2\}$, $\{D_3, D_4\}$, $\{D_1 \cup D_2, D_3 \cup D_4\}$ are domain decompositions of $D_1 \cup D_2$, $D_3 \cup D_4$, Ω respectively, which satisfy Condition(A) for $p = 2$.

Condition(C). The subregion $\Omega_i (1 \le i \le p)$ can be devided into k parts:

$$D_j = \sum_{r_{j-1}+1 \le i \le r_j} \Omega_i, \ j = 1, 2, ..., k, \ r_0 = 0, \ r_k = p, \tag{11}$$

such that: (1) $\{\Omega_j, r_{j-1} + 1 \le i \le r_j\}$ is a domain decomposition of D_j satisfying condition(A); (2) for $r_{j-1} + 1 \le i, l \le r_j, \Omega_i \cap \Omega_l = \emptyset$, if $l \ne i - 1, i + 1$; (3) $\{D_j, 1 \le j \le k\}$ is a domain decomposition of Ω satisfying condition(A); (4) for $1 \le j, l \le k$, $D_j \cap D_l = \emptyset$, if $l \ne j - 1, j + 1$.

Now we give a example satisfying Condition (A) or (B). We divide a square into 16 subregions. Condition (B) holds for the subregion order listed in Figure 1 and Condition (C) holds for the subregion order listed in Figure 2.

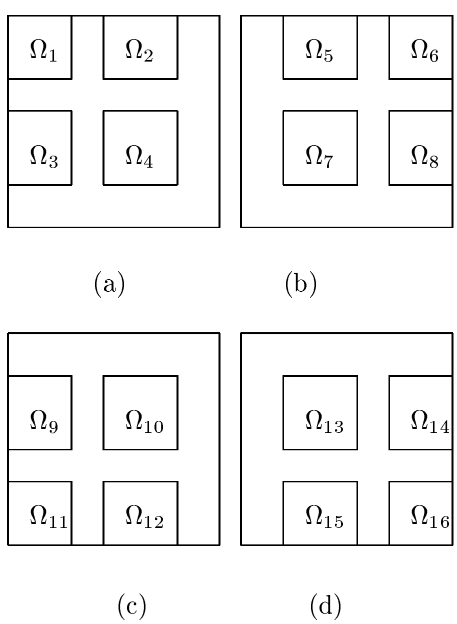

Figure 1

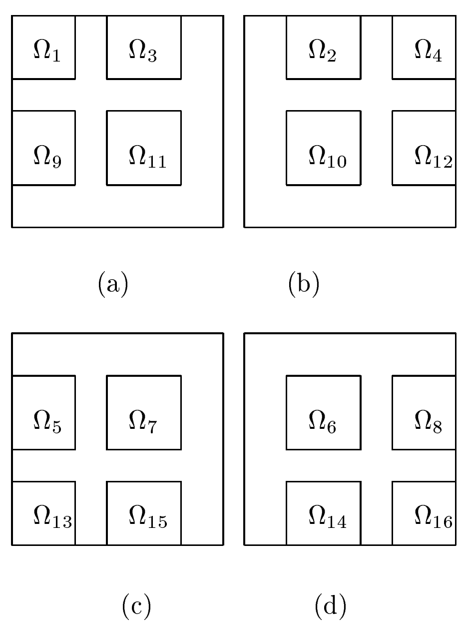

Figure 2

Extending the elements in $H_0^1(\Omega_j)$ to Ω by zero, we give two kinds of Schwarz type domain decomposition algorithm. First we give the semi-discreted Schwarz algorithms.

Scheme I. Let $U^0 = u^0$, for $n \geq 1$ we find $U^n \in H_0^1(\Omega)$ by three steps:

1) Set $U_0^n = U^{n-1}$;

2) Find $U_{jp+i}^n (j = 0, 1, ..., m-1, i = 1, 2, ..., p)$ such that

$$(\frac{U_{jp+i}^n - \bar{U}^{n-1}}{\triangle t}, v) + a(U_{jp+i}^n, v) = (f(U^{n-1}), v) \quad v \in H_0^1(\Omega_i), \tag{12}$$

$$U_{jp+i}^n = U_{jp+i-1}^n \quad x \in \Omega \setminus \Omega_i; \tag{13}$$

3) Let $U^n = U_{mp}^n \quad x \in \Omega$.

Here m denote the iteration number at every time-level.

Subdomain problems (12)-(13) are continuous in spacial variable, in practical computation we can use appropriate numerical method to solve them. Next we shall give a kind of multiplicative Schwarz algorithm combained with Galerkin finite element method.

Let T_h denote a quasi-uniform triangulation of Ω which is obtained by refining the above domain decomposition, h is the meshsize. $M_h \subset H_0^1(\Omega)$ denote the standard finite element space such that

$$\inf_{\varphi \in M_h} (\|u - \varphi\| + h\|u - \varphi\|_1) \leq C\|u\|_{k+1} h^{k+1} \quad u \in H_0^1(\Omega) \cap H^{k+1}(\Omega). \tag{14}$$

Let $T_{i,h}$ denote the restriction of T_h on Ω_i, $M_h(\Omega_i)$ denote the restriction of M_h on Ω_i, $M_h^0(\Omega_i) = M^h(\Omega_i) \cap H_0^1(\Omega_i)$. Set the initial approximation $W_h^0 \in M_h$ satisfying

$$a(W_h^0, v) = a(u^0, v), \quad \forall v \in M_h \tag{15}$$

We give the following multiplicative Schwarz algorithm:

Scheme II. For $n \geq 1$ find$W^n \in M_h$ by three steps:

1) Set $W_0^n = W^{n-1}$;

2) Find $W_{jp+i}^n (j = 0, 1, ..., m-1, i = 1, 2, ..., p)$ satisfying

$$(\frac{W_{jp+i}^n - \bar{W}^{n-1}}{\triangle t}, v) + a(W_{jp+i}^n, v) = (f(W^{n-1}), v), \quad v \in M_h^0(\Omega_i), \tag{16}$$

$$W_{jp+i}^n = W_{jp+i-1}^n, \quad x \in \Omega \setminus \Omega_i; \tag{17}$$

3) Let $W^n = W_{mp}^n, x \in \Omega$.

Here m denote the iteration number at every time-level.

It is clear that the solutions of the above schemes are defined uniquely.

Remark 1. When Condition (B) holds the above method can be parallelized by coloring the subdomains and solving simultaneously on disjoint subdomains of the same color.

let

$$A(u, v) = (u, v) + \triangle t a(u, v) \quad \forall v \in H_0^1(\Omega), \tag{18}$$

$$\|u\|_a = (a(u, u))^{\frac{1}{2}}, \tag{19}$$

$$\|u\|_A = (A(u, u))^{\frac{1}{2}} = (\|u\|_0^2 + \|u\|_a^2)^{\frac{1}{2}}. \tag{20}$$

In Section 4 we can prove the following main results.

Theorem 2.1. *Suppose that the solution of (1) is sufficiently smooth. Suppose also that Condition(B) or Condition(C) holds for the above domain decomposition. When $m \geq 1$ for the solution of Scheme I we have that*

$$\|u^n - U^n\| \leq C(\triangle t + \triangle t \|\frac{\partial^2 u}{\partial \tau^2}\|_{L^\infty(J; L^2(\Omega))} + \triangle t^{\frac{m}{2}}), \tag{21}$$

where C denotes a generic constant independent of $\triangle t$.

Theorem 2.2. *Under the assumptions of Theorem 2.1. When $m \geq 1$, $h^m = O(\Delta t)$ for the solution of Scheme II we have that*

$$\|u^n - W^n\| \leq C(h^{k+1} + \triangle t + \triangle t\|\frac{\partial^2 u}{\partial \tau^2}\|_{L^\infty(J;L^2(\Omega))} + (\triangle t + h)^{\frac{m}{2}}), \tag{22}$$

where C denotes a generic constant independent of $\triangle t$ and h.

Theorems 2.1, 2.2 give the optimal order estimates when $m = 2$, and $m = 2(k+1)$, respectively, while in [13] we only considered the linear, symmetric parabolic problem.

3. Some Lemmas

In this section we give some lemmas under Condition (A), (B) or (C). For simplicity we omit their proofs, because they are similar to the lemmas given in [13]. See [13] for detail.

Lemma 3.1. *Suppose the domain decomposition satisfy Condition (A), then for $u \in H_0^1(\Omega)$ there exists a decomposition $u = \sum_{l=1}^{p} u_l$, $u_l \in H_0^1(\Omega_l)$ such that*

$$\sum_{l=1}^{p} \|u_l\|_A^2 \leq (1 + C_1\triangle t)\|u\|_A^2, \tag{23}$$

where C_1 denotes a constant independent of $\triangle t$ and u.

Lemma 3.2. *Under the condition of Lemma 3.1, for $u \in M_h$ there exists a decomposition $u = \sum_{l=1}^{p} u_l$, $u_l \in M_h^0(\Omega_l)$ such that*

$$\sum_{l=1}^{p} \|u_l\|_A^2 \leq (1 + C_2(\triangle t + h))\|u\|_A^2, \tag{24}$$

where C_2 denotes a constant independent of h, $\triangle t$ and u.

Defining an operator R_i: $H_0^1(\Omega) \to H_0^1(\Omega_i)$ such that

$$A(R_i u - u, v) = 0 \quad \forall v \in H_0^1(\Omega_i) \tag{25}$$

Also define $R_{h,i}$: $M_h \to M_h^0(\Omega_i)$ such that

$$A(R_{h,i} u - u, v) = 0 \quad \forall v \in M_h^0(\Omega_i). \tag{26}$$

It is clear that $R_i, R_{i,h}$ is bounded,

$$\|R_i u\|_A \leq \|u\|_A \quad u \in H_0^1(\Omega_i), \quad \|R_{h,i} u\|_A \leq \|u\|_A \quad u \in M_h^0(\Omega_i). \tag{27}$$

Using the above lemmas, similarly to [13] we can prove the following lemmas.

Lemma 3.3. *Suppose Condition (B) or Condition (C) hold for the domain decomposition, then there exists a constant C_3 independent of u, $\triangle t$ such that*

$$\|(I - R_p)\cdots(I - R_2)(I - R_1)u\|_A \leq C_3\triangle t^{\frac{1}{2}}\|u\|_A \quad \forall u \in H_0^1(\Omega). \tag{28}$$

Lemma 3.4. *Suppose Condition (B) or condition (C) be satisfied for the domain decomposition. Then there exists a constant C_4 independent of u, h, $\triangle t$ such that*

$$\|(I - R_{h,p})\cdots(I - R_{h,2})(I - R_{h,1})u\|_A \leq C_4(h + \triangle t)^{\frac{1}{2}}\|u\|_A, \quad \forall u \in M_h. \tag{29}$$

Lemmas 3.3, 3.4 tell us that the convergence rates for Scheme I and II are $C_3\triangle t^{\frac{1}{2}}$ and $C_4(h+\triangle t)^{\frac{1}{2}}$ respectively, which become small when the discretization parameters become small.

4. Convergence Analyses and Error Estimates

In this section we give the error estimates for Algorithm II, that is the proof of Theorem 2. The analyses of Scheme I can be given similarly. Define the auxiliary project $u_h \in M_h$ such that

$$a(u_h - u, v) = 0 \quad \forall v \in M_h. \tag{30}$$

Let $\eta = u_h - u$, then

$$\|\eta^n\|_s + h\|\frac{\partial \eta}{\partial t}\|_s \le Ch^{k+1-s}, s = 0, 1. \tag{31}$$

Using (9) we derive that

$$(\frac{u_h^n - \bar{u}_h^{n-1}}{\Delta t}, v) + a(u_h^n, v) = (\partial_t \eta^n + \rho^n, v) + (\frac{\eta^{n-1} - \bar{\eta}^{n-1}}{\Delta t}, v) + (f(u^{n-1}), v) \quad \forall v \in M_h. \tag{32}$$

Let $e^n = W^n - u_h^n$, $e_i^n = W_i^n - u_h^n$, then $e^n = e_{mp}^n$ and combining (16) and (32) we have that for any $v \in M_h^0(\Omega_i)$

$$(\frac{e_{jp+i}^n - \bar{e}^{n-1}}{\triangle t}, v) + a(e_{jp+i}^n, v) = (f' \cdot (e^{n-1} + \eta^{n-1}) - \partial_t \eta^n - \rho^n - \frac{\eta^{n-1} - \bar{\eta}^{n-1}}{\triangle t}, v), \tag{33}$$

$$e_{jp+i}^n - e_{jp+i-1}^n = 0 \quad x \in \Omega - \Omega_i, \tag{34}$$

where $f' \cdot (e^{n-1} + \eta^{n-1}) = f(W^{n-1}) - f(u^{n-1})$, f' denotes the first order derivative of f at a point between W^{n-1} and u^{n-1}. This can be changed into

$$\begin{aligned} A(e_{jp+i}^n, v) = (e_{jp+i}^n, v) + \triangle t a(e^n_, v) = (\bar{e}^{n-1}, v) + \Delta t(f' \cdot (e^{n-1} + \eta^{n-1}), v) \\ -\Delta t(\partial_t \eta^n + \rho^n, v) - (\eta^{n-1} - \bar{\eta}^{n-1}, v) \quad \forall v \in M_h^0(\Omega_i) \end{aligned} \tag{35}$$

Let $E^n \in M_h$ be the solution of the equation

$$\begin{aligned} (E^n, v) + \triangle t a(E^n, v) = (\bar{e}^{n-1}, v) + \Delta t(f' \cdot (e^{n-1} + \eta^{n-1}), v) - \Delta t(\partial_t \eta^n + \rho^n, v) \\ -(\eta^{n-1} - \bar{\eta}^{n-1}, v) \quad \forall v \in M_h. \end{aligned} \tag{36}$$

From (35), (36) we have that

$$A(e_{jp+i}^n - E^n - (e_{jp+i-1}^n - E^n), v) = -A(e_{jp+i-1}^n - E^n, v) \quad \forall v \in M_h^0(\Omega_i) \tag{37}$$

Since $e_{jp+i}^n - E^n - (e_{jp+i-1}^n - E^n) \in M_h^0(\Omega_i)$, we have

$$e_{jp+i}^n - E^n - (e_{jp+i-1}^n - E^n) = -R_{h,i}(e_{jp+i-1}^n - E^n).$$

$$e_{jp+i}^n - E^n = (I - R_{h,i})(e_{jp+i-1}^n - E^n), \ i = 1, 2, ..., p. \tag{38}$$

Therefore when condition (B) or condition (C) holds we have

$$e^n - E^n = (e_{mp}^n - E^n) = ((I - R_{h,p}) \cdots (I - R_{h,1}))^m (e_0^n - E^n). \tag{39}$$

$$\begin{aligned} \|e^n - E^n\|_A &\le \|((I - R_{h,p}) \cdots (I - R_{h,1}))^m (e_0^n - E^n)\|_A \\ &\le C(\triangle t + h)^{\frac{m}{2}} \|W^{n-1} - u_h^{n-1} + u_h^{n-1} - u_h^n - E^n\|_A, \\ \|e^n\|_A &\le \|E^n\|_A + C(\triangle t + h)^{\frac{m}{2}} \|W^{n-1} - E^n + u_h^{n-1} - u_h^n\|_A, \end{aligned} \tag{40}$$

noticing $\|u_h^n - u_h^{n-1}\|_A^2 = O(\triangle t \int_{t^{n-1}}^{t^n} \|\frac{\partial u_h}{\partial t}\|_A^2 dt)$, we have that

$$\|e^n\|_A^2 \le (1 + C\triangle t)\|E^n\|_A^2 + [\epsilon \frac{(\triangle t + h)^m}{\triangle t} + (\triangle t + h)^m]\|e^{n-1} - E^n\|_A^2$$
$$+C(\triangle t + h)^m \int_{t^{n-1}}^{t^n} \|\frac{\partial u_h}{\partial t}\|_A^2 dt, \tag{41}$$

where the constant ϵ will be determined later.

Now we estimate the right-hand terms of (41). Let $v = E^n$ in (36) we have that

$$\|E^n\|_A^2 = (e^{n-1}, E^n) + (\bar{e}^{n-1} - e^{n-1}, E^n) + \Delta t(f' \cdot (e^{n-1} + \eta^{n-1}), E^n)$$
$$-\Delta t(\partial_t \eta^n + \rho^n, E^n) - (\eta^{n-1} - \bar{\eta}^{n-1}, E^n), \quad \forall v \in M_h.$$
$$\le \|e^{n-1}\|\|E^n\| + C\triangle t\|e^{n-1}\|\|\nabla E^n\| + C\Delta t\|e^{n-1}\|\|E^n\|$$
$$+\Delta t(\|\partial_t \eta^n\| + \|\rho^n\|)\|E^n\| + C\Delta t\|\eta^{n-1}\|\|\nabla E^n\|$$
$$\le \frac{1}{2}\|E^n\|_A^2 + \frac{1}{2}(1 + C\triangle t)\|e^{n-1}\|^2$$
$$+C\triangle t(\|\partial_t \eta^n\|^2 + \|\rho^n\|^2 + \|\eta^{n-1}\|^2). \tag{42}$$

Using the result of [8, 12, 14] we have that

$$\|\bar{e}^{n-1}\| \le \|e^{n-1}\| + \|\bar{e}^{n-1} - e^{n-1}\| \le \|e^{n-1}\| + C\triangle t\|e^{n-1}\|_a, \tag{43}$$

$$(\bar{\eta}^{n-1} - \eta^{n-1}, E^n) \le C\|\eta^{n-1}\| \cdot \|E^n\|_a \Delta t \le C\Delta t^{\frac{1}{2}} h^{k+1}\|E^n\|_A, \tag{44}$$

therefore

$$\|E^n\|_A^2 \le (1 + C\Delta t)\|e^{n-1}\|^2 + C\Delta t(\|\partial_t \eta^n\|^2 + \|\rho^n\|^2 + \|\eta^{n-1}\|^2). \tag{45}$$

From (36) we have that

$$(E^n - e^{n-1}, v) + \triangle t a(E^n - e^{n-1}, v) = (\bar{e}^{n-1} - e^{n-1}, v) + \Delta t(f' \cdot (e^{n-1} + \eta^{n-1}), v)$$
$$-\Delta t(\partial_t \eta^n + \rho^n, v) - (\eta^{n-1} - \bar{\eta}^{n-1}, v) - \triangle t a(e^{n-1}, E^n - e^{n-1}) \quad \forall v \in M_h. \tag{46}$$

Let $v = E^n - e^{n-1}$, using the same method as (45) we have that

$$\|E^n - e^{n-1}\|_A^2 = \|E^n - e^{n-1}\|^2 + \triangle t\|E^n - e^{n-1}\|_a^2$$
$$= (\bar{e}^{n-1} - e^{n-1}, E^n - e^{n-1}) + \Delta t(f' \cdot (e^{n-1} + \eta^{n-1}), E^n - e^{n-1}) - \Delta t(\partial_t \eta^n + \rho^n, E^n - e^{n-1})$$
$$-(\eta^{n-1} - \bar{\eta}^{n-1}, E^n - e^{n-1}) - \triangle t a(e^{n-1}, E^n - e^{n-1})$$
$$\le C[\Delta t\|e^{n-1}\|_a\|E^n - e^{n-1}\| + \Delta t\|e^{n-1}\|\|E^n - e^{n-1}\| + \Delta t(\|\partial_t \eta^n\| + \|\rho^n\|)\|E^n - e^{n-1}\|$$
$$+C\Delta t\|\eta^{n-1}\|\|E^n - e^{n-1}\|_a + \triangle t\|e^{n-1}\|_a\|E^n - e^{n-1}\|_a]$$
$$\le \frac{1}{2}\|E^n - e^{n-1}\|_A^2 + C\Delta t[\|e^{n-1}\|_a^2 + \|e^{n-1}\|^2 + \Delta t(\|\partial_t \eta^n\|^2 + \|\rho^n\|^2) + \|\eta^{n-1}\|^2], \tag{47}$$

therefore

$$\|E^n - e^{n-1}\|_A^2 \le C\Delta t[\|e^{n-1}\|^2 + \|e^{n-1}\|_a^2 + \Delta t(\|\partial_t \eta^n\|^2 + \|\rho^n\|^2) + \|\eta^{n-1}\|^2]. \tag{48}$$

Using (45), (48) and (41) we have that

$$\|e^n\|_A^2 \le (1 + C\Delta t)\|e^{n-1}\|^2 + C(\epsilon + \Delta t)(\Delta t + h)^m(\|e^{n-1}\|_a + \|e^{n-1}\|^2)$$

$$+C[\Delta t(\|\partial_t\eta^n\|^2+\|\rho^n\|^2)+(\Delta t+(\Delta t+h)^m)\|\eta^{n-1}\|^2+(\Delta t+h)^m\int_{t^{n-1}}^{t^n}\|\frac{\partial u_h}{\partial t}\|^2dt].$$

When $h^m=O(\triangle t)$, it becomes

$$\|e^n\|^2+\triangle t\|e^n\|_a^2\le(1+C\Delta t)\|e^{n-1}\|^2+C(\epsilon+\Delta t)\Delta t\|e^{n-1}\|_a^2$$

$$+C\triangle t[\Delta t(\|\partial_t\eta^n\|^2+\|\rho^n\|^2)+\|\eta^{n-1}\|^2]+(\Delta t+h)^m\int_{t^{n-1}}^{t^n}\|\frac{\partial u_h}{\partial t}\|^2dt]. \quad (49)$$

Make summation for n and let $\epsilon=\frac{1}{4}$, when $\Delta t\le\frac{1}{4}$ we have

$$\|e^n\|^2+\sum_{j=1}^n\triangle t\|e^n\|_a^2\le C[\|e^0\|^2+\Delta t\|e^0\|_a^2+h^{2(k+1)}\int_0^T(\|u\|_{H^{k+1}}^2+\|\frac{\partial u}{\partial t}\|_{H^{k+1}}^2)dt$$

$$+(\Delta t+h)^m\int_0^T\|\frac{\partial u_h}{\partial t}\|_a^2dt+\Delta t^2\int_0^T\|\frac{\partial u_h}{\partial\tau}\|^2dt].$$

Since $e^0=0$,

$$W^n-u^n=W^n-u_h^n+u_h^n-u^n=e^n+\eta^n.$$

We have that

$$\|W^n-u^n\|\le\|e^n\|+\|\eta^n\|$$

$$\le C(\triangle t+\Delta t\|\frac{\partial^2u}{\partial\tau^2}\|_{L^2(J;L^2)}+h^{k+1}+(\triangle t+h)^{\frac{m}{2}}). \quad (50)$$

That completes the proof of Theorem 2.2.

Similarly using Lemma 3.3 we can prove Theorem 2.1.

5. Numerical Examples

Consider the following convection diffusion equation,

$$\frac{\partial u}{\partial t}+b\cdot\triangledown u-\nabla(D\nabla u)=f,\ \text{in}\ \Omega\times J, \quad (51)$$

where $\Omega=(0,1)\times(0,1)$, $J=(0,1)$, $b=(1,1)$, $D=dI$, where I is the unit matrix, the coefficient $d>0$. The righthand term, boundary condition and initial condition are selected in such a way that the exact solution is $u=exp(x+y-t)$. We divide Ω into four subdomains: $\Omega_1=(0,0.6)*(0,0.6)$, $\Omega_2=(0.4,1)*(0,0.6)$, $\Omega_3=(0,0.6)*(0.4,1)$, $\Omega_4=(0.4,1)*(0.4,1)$, see Figure 3.

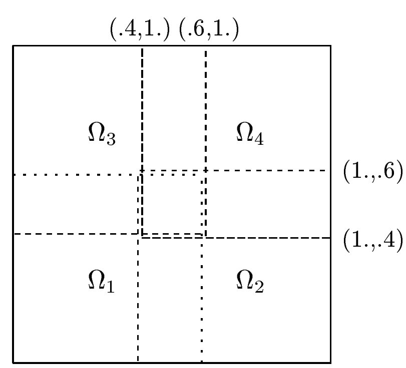

Figure 3: The Subdomains of Ω

Based on this domain decomposition we give the triangulation. We first divide Ω into uniform squares with meshsize h, here we let $h = 0.1, 0.05, 0.025$, then we obtain the triangulation by dividing each square into two triangles. The time increment is $\triangle t$. The iterative number in each time step is M.

Define the the discrete L^2 norm error and L_∞ norm error as follow,

$$E_2 = \max_{1\le n\le \frac{T}{\triangle t}} \Big(\sum_{0<i,j<\frac{1}{h}} (u^n_{h,ij} - u^n_{ij})^2 h^2 \Big)^{\frac{1}{2}},$$

$$E_\infty = \max_{1\le n\le \frac{T}{\triangle t}} \Big(\max_{0<i,j<\frac{1}{h}} |u^n_{h,ij} - u^n_{ij}| \Big),$$

where $u^n_{ij}, u^n_{h,ij}$ denote the exact solution and approximate solution at point $(ih, jh, n\triangle t)$, respectively.

Setting $d = 1., 0.01$, using different parameters $\triangle t$, h and M, the errors E_2, E_∞ are depicted in the following table.

			d=1.0		d=0.01	
$\triangle t$	h	M	L_2	L_∞	L_2	L_∞
0.10	0.10	1	6.987E-2	1.575E-1	2.1797E-2	5.4174E-2
0.10	0.10	2	1.883E-2	4.555E-2	2.0735E-2	5.2585E-2
0.10	0.10	3	8.177E-3	1.498E-2	2.0733E-2	5.2578E-2
0.10	0.10	4	5.545E-3	6.360E-3	2.0733E-2	5.2578E-2
0.05	0.05	1	3.045E-2	6.852E-2	1.0864E-2	3.0157E-2
0.05	0.05	2	6.736E-3	1.662E-2	1.0826E-2	3.0075E-2
0.05	0.05	3	2.610E-3	4.655E-3	1.0826E-2	3.0075E-2
0.05	0.05	4	1.839E-3	2.359E-3	1.0826E-2	3.0075E-2
0.025	0.025	1	1.311E-2	2.955E-2	5.5437E-2	1.6174E-2
0.025	0.025	2	2.156E-3	5.208E-3	5.5433E-2	1.6173E-2
0.025	0.025	3	8.339E-4	1.312E-3	5.5433E-2	1.6173E-2
0.025	0.025	4	6.886E-4	6.033E-3	5.5433E-2	1.6173E-2

From the numerical results we can see that using the multiplicative Schwarz method we can get a good result for convection diffusion problem, even iterating only one or two cycle at each time level. When the diffusion coefficient is small, the convection term is dominated, which is dealt with by characteristic method, so in the numerical examples there are little diffenences between $M = 1, 2, 3, 4$.

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