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MONOTONE APPROXIMATION TO A SYSTEM WITHOUT MONOTONE NONLINEARITY*

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Abstract

A monotone approximation is proposed for a system without monotone nonlinearity. A new concept of ordered pair of supersolution and subsolution is introduced, and then the existence of numerical solutions is studied. A new monotone iteration is provided for solving the resulting problem. An approximation with high accuracy is investigated. The corresponding iteration possesses geometric convergence rate. The numerical results support the theoretical analysis.

Key words: Monotone approximation, Systems without monotone nonlinearity.

1. Introduction

Due to the development of various studies in electromagnetism, biology and some other fields, nonlinear systems have been paid extensive attention both analytically and numerically, e.g., see [1–12]. As we know, a reasonable numerical method should not only have the approximation error of higher order, but also preserve the main feature of the original problem. In this case, the numerical results might fit the physical process better. Since the usual approximations simulate the maximum principle, they are of positive-type. Thus they possess only the second order. In [13], the authors proposed a new approach for a nonlinear equation. This approach simulates the comparison principle and thus provides the higher accuracy. Later, the authors generalized this approach to nonlinear systems, e.g., see [14]. However, the corresponding analysis is valid only when the nonlinear terms are monotone in some sense. This fact limits the application of this new approach. So, the question whether it is possible to develop this approach for some systems without any monotone nonlinearity, is natural and interesting. In this paper we investigate this problem. The answer is positive.

The outline of this paper is as follows. In Section 2, we present the monotone approximation for a system without any monotone nonlinearity. Then we introduce a new concept of ordered pair of supersolution and subsolution for the resulting problem, and study the existence of numerical solutions. Finally, we propose a new iteration for solving the resulting problem. In Section 3, we investigate a monotone approximation on uniform mesh. Especially, we give a sufficient condition ensuring the convergence

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of iteration to the unique solution of the corresponding problem in some cone. We also show that the iteration has the geometric convergence rate under some reasonable conditions. In the final section, we list the numerical results which coincide with the theoretical analysis in the previous sections, and show the advantages of this method.

2. General Framework

Let $I = \{x|\ 0 < x < 1\}$ and $\overline{I}$ be its closure. $u = (u_1, u_2, \cdots, u_n)^T$ denotes a vector function of x. The given function $f(x,u) \in [C^0(\overline{I} \times \mathbf{R}^n) \cap C^1(I \times \mathbf{R}^n)]^n$ has the components $f_i(x,u)$. Also let $a_i(x) \in C^1(I)$ and assume that for certain positive constants $\alpha_0 \le \alpha_1$, $\alpha_0 \le a_i(x) \le \alpha_1$ for $x \in I$ and $1 \le i \le n$. Moreover suppose that $\left|\frac{\mathrm{d}a_i}{\mathrm{d}x}(x)\right|$ is bounded for $x \in I$ and $1 \le i \le n$. Furthermore, let $l = \mathrm{diag}(l_1, \cdots, l_n)$ with

$$l_i u_i(x) = -(a_i(x)u_i'(x))', \quad u_i'(x) = \frac{\mathrm{d}u_i}{\mathrm{d}x}(x), \qquad 1 \le i \le n.$$

Set $F_{i,j}(x,u) = \frac{\partial f_i}{\partial u_j}(x,u)$, $1 \le i,j \le n$. We consider the following coupled problem, i.e., finding $u(x) \in [C^0(\overline{I}) \cap C^2(I)]^n$ such that

$$\begin{cases} lu(x) + f(x,u(x)) = 0, & x \in I, \\ u(0) = u(1) = 0. \end{cases} \tag{2.1}$$

Such a problem arises in many fields, e.g., see [15]. It is well known that if $u(x) \in [C^0(\overline{I}) \cap C^2(I)]^n$ and

$$\begin{cases} lu(x) \ge 0, & x \in I, \\ u(0) \ge 0,\ u(1) \ge 0, \end{cases}$$

then $u(x) \ge 0$ for $x \in \overline{I}$. Conversely, if the reversed inequalities hold in the above problem, then $u(x) \le 0$ for $x \in \overline{I}$.

It is not difficult to prove the existence of solutions of (2.1) under some conditions. To solve (2.1) numerically, we introduce a set of mesh points $\{x_p\}_0^N$ such that

$$0 = x_0 < x_1 < \cdots < x_{N-1} < x_N = 1.$$

For each p, let $I_p = (x_{p-1}, x_p)$, $\overline{I}_p = [x_{p-1}, x_p]$, $h_p = x_p - x_{p-1}$, and $h = \max_{1 \le p \le N} h_p$. We also set $I_h = \{x_p\}_1^{N-1}$ and $\overline{I}_h = \{x_p\}_0^N$. Next let $l_h = \mathrm{diag}(l_{h,1}, \cdots, l_{h.n})$ and $P_h = \mathrm{diag}(P_{h,1}, \cdots, P_{h.n})$ be certain linear discrete operators. Then the corresponding discrete problem might be stated as follows, i.e., finding $u_h(x) = (u_{h.1}(x), \cdots, u_{h,n}(x))^T$ such that

$$\begin{cases} l_h u_h(x) + P_h f(x, u_h(x)) = 0, & x \in I_h, \\ u_h(0) = u_h(1) = 0. \end{cases} \tag{2.2}$$

We say that (2.2) is a monotone approximation, if the following conditions are fulfilled,

($\mathbf{H_1}$) There exists a positive constant h_0 such that for all $h \le h_0$, the operator P_h is positive, that is , if $q(x) = (q_1(x), \cdots, q_n(x))^T \ge 0$, then $P_h q(x) \ge 0$ for all $x \in \overline{I}_h$.

($\mathbf{H_2}$) There exists a positive constant h_1 such that for all $h \le h_1$, the system

$$\begin{cases} l_h u_h(x) \ge 0, & x \in I_h, \\ u_h(0) \ge 0,\ u_h(1) \ge 0 \end{cases}$$

implies that $u_h(x) \ge 0$ for all $x \in \overline{I}_h$.

Hereafter, we suppose that (2.2) is a monotone approximation with the constants h_0 and h_1. Clearly for all $h \le h_1$, the linear problem

$$\begin{cases} l_h u_h(x) = g(x), & x \in I_h, \\ u_h(0) = u_h(1) = 0 \end{cases}$$

is uniquely solvable. The operator l_h keeps the comparison principles in the following sense.

Lemma 2.1. *If $h \le h_1$ and*

$$\begin{cases} l_h u_h(x) \ge 0, & x \in I_h, \\ u_h(0) \ge 0,\ u_h(1) \ge 0, \end{cases}$$

then $u_h(x) \ge 0$ for all $x \in \overline{I}_h$. Similarly, if the reversed inequalities hold for all $h \le h_1$, then $u_h(x) \le 0$ for all $x \in \overline{I}_h$.

Furthermore, if $f(x, u) \le 0$ in $I \times \mathbf{R}^n$ and $h \le \min(h_0, h_1)$, then all possible solutions of (2.2) are nonnegative.

Without further mention, we assume that all the inequalities involving vectors are componentwise. Let $u_h(x)$, $u_{h*}(x)$ and $u_h^*(x)$ be given vector functions on $\overline{I}_h$. If $u_{h*}(x) \le u_h(x) \le u_h^*(x)$ on $\overline{I}_h$, we say that $u_h \in \mathbf{K}(u_{h*}, u_h^*)$.

We now introduce a new concept of ordered pair of supersolution and subsolution for (2.2), which is the generalized discrete simulation of those for continuous version as in [6, 7], and the generalization of the corresponding discrete version as in [13, 14].

Definition 2.1. *A pair of vector functions $\overline{u}_h(x)$ and $\underline{u}_h(x)$ is called an ordered pair of supersolution and subsolution for (2.2), if $\underline{u}_h(x) \le \overline{u}_h(x)$ on $\overline{I}_h$ and there exists a nonnegative matrix $B = (B_{i,j})$ such that*

(i) *for all $\underline{u}_h(x) \le v_h(x) \le u_h(x) \le \overline{u}_h(x)$ and $x \in I_h$,*

$$-B(u_h(x) - v_h(x)) \le f(x, u_h(x)) - f(x, v_h(x)) \le B(u_h(x) - v_h(x));$$

(ii)

$$\begin{cases} l_h \overline{u}_h(x) + P_h f(x, \overline{u}_h(x)) \ge P_h\left(B(\overline{u}_h(x) - \underline{u}_h(x))\right), & x \in I_h, \\ l_h \underline{u}_h(x) + P_h f(x, \underline{u}_h(x)) \le -P_h\left(B(\overline{u}_h(x) - \underline{u}_h(x))\right), & x \in I_h, \\ \overline{u}_h(x) \ge 0, \quad \underline{u}_h(x) \le 0, & x = 0, 1. \end{cases}$$

Remark 2.1. The above definition does not need any monotonicity for $f(x,u)$. But the usual definition is only for the system in which $f(x,u)$ fulfills certain monotonicity.

Theorem 2.1. *Suppose that $\{\overline{u}_h, \underline{u}_h\}$ is an ordered pair of supersolution and subsolution for (2.2), and $h \le \min(h_0, h_1)$. Then problem (2.2) has at least one solution $u_h \in \mathbf{K}(\underline{u}_h, \overline{u}_h)$.*

Proof. We consider the following auxiliary system

$$\begin{cases} l_h u_h(x) = -P_h f(x, v_h(x)), & x \in I_h, \\ u_h(0) = u_h(1) = 0. \end{cases} \tag{2.3}$$

For all $h \le \min(h_0, h_1)$ and any $v_h \in \mathbf{K}(\underline{u}_h, \overline{u}_h)$, the condition $(\mathbf{H_2})$ implies the existence and uniqueness of the solution $u_h(x)$ of (2.3). Define the map T on $\mathbf{K}(\underline{u}_h, \overline{u}_h)$ as

$$Tv_h = u_h, \qquad \forall v_h \in \mathbf{K}(\underline{u}_h, \overline{u}_h). \tag{2.4}$$

We first show that $u_h \in \mathbf{K}(\underline{u}_h, \overline{u}_h)$. Let $w_h(x) = u_h(x) - \underline{u}_h(x)$. By definition 2.1,

$$\begin{cases} l_h w_h(x) = -P_h f(x, v_h(x)) - l_h \underline{u}_h(x) \\ \qquad \ge -P_h f(x, v_h(x)) + P_h\left(B(\overline{u}_h(x) - \underline{u}_h(x))\right) + P_h f(x, \underline{u}_h(x)) \\ \qquad \ge P_h\left(B(\overline{u}_h(x) - \underline{u}_h(x))\right) - P_h\left(B(v_h(x) - \underline{u}_h(x))\right) \\ \qquad \ge 0, \qquad x \in I_h, \\ w_h(0) \ge 0, \quad w_h(1) \ge 0. \end{cases}$$

By the condition $(\mathbf{H_2})$, $w_h(x) \ge 0$ and so for all $x \in \overline{I}_h$, $u_h(x) \ge \underline{u}_h(x)$. Similarly, for all $x \in \overline{I}_h$, $u_h(x) \le \overline{u}_h(x)$.

We next show that T is a bounded continuous map from $\mathbf{K}(\underline{u}_h, \overline{u}_h)$ into itself. Clearly, T is bounded. Let $v_h^{(m)} \in \mathbf{K}(\underline{u}_h, \overline{u}_h)$ be a sequence converging to $v_h(x)$. Since $\mathbf{K}(\underline{u}_h, \overline{u}_h)$ is a closed convex cone, we have $v_h \in \mathbf{K}(\underline{u}_h, \overline{u}_h)$. Define

$$Tv_h^{(m)} = u_h^{(m)}, \qquad Tv_h = u_h.$$

Since the sequence $\left\{u_h^{(m)}(x)\right\}$ is bounded, we can select a subsequence $\left\{u_h^{(m_k)}(x)\right\}$ such that for all $x \in \overline{I}_h$, $\left\{u_h^{(m_k)}(x)\right\}$ converges to a limit, say, $u_h^*(x)$. By the definition,

$$\begin{cases} l_h u_h^{(m_k)}(x) = -P_h f(x, v_h^{(m_k)}(x)), & x \in I_h, \\ u_h^{(m_k)}(0) = u_h^{(m_k)}(1) = 0. \end{cases}$$

Letting $k \to \infty$, we obtain

$$\begin{cases} l_h u_h^*(x) = -P_h f(x, v_h(x)), & x \in I_h, \\ u_h^*(0) = u_h^*(1) = 0. \end{cases}$$

Thus both $u_h^*(x)$ and $u_h(x)$ are solutions of the same linear problem. By the uniqueness of the solution, $u_h^*(x) = u_h(x)$ for all $x \in \overline{I}_h$. Hence $\left\{u_h^{(m_k)}(x)\right\}$ converges to $u_h(x)$ as $k \to \infty$. Finally, we claim that the whole sequence $\left\{u_h^{(m)}(x)\right\}$ converges to $u_h(x)$ as $m \to \infty$. Indeed, if it is not so, then there exists a subsequence $\{u_h^{(m_j)}(x)\}$ and a constant $\varepsilon_0 > 0$ such that

$$\max_i \max_{x \in I_h} \left| u_{h,i}^{(m_j)}(x) - u_{h,i}(x) \right| \geq \varepsilon_0, \qquad j = 1, 2, \cdots. \tag{2.5}$$

On the other hand, by the same argument as before, we can select a subsequence of $\left\{u_h^{(m_j)}(x)\right\}$ converging to $u_h(x)$ as $j \to \infty$. This contradicts inequality (2.5). Consequently, the sequence $\left\{u_h^{(m)}(x)\right\}$ converges to $u_h(x)$ as $m \to \infty$. This fact indicates that T is a continuous map from $\mathbf{K}(\underline{u}_h, \overline{u}_h)$ into itself. Furthermore, since $\mathbf{K}(\underline{u}_h, \overline{u}_h)$ is a finite dimensional space, T is a completely continuous map. Thus by Schauder's fixed point theorem, T has at least one fixed point $u_h^* \in \mathbf{K}(\underline{u}_h, \overline{u}_h)$. This is a solution of problem (2.2).

Theorem 2.1 shows that if (2.2) possesses an ordered pair of supersolution and subsolution, then it has at least one solution. Moreover, the supersolutions and the subsolutions may serve as the upper bounds and the lower bounds for the solutions. In the coming part, we shall propose a modified monotone iteration which improves the bounds. Besides, if the sequences of the upper bounds and the lower bounds converge to the same limit, then it forms the unique solution of problem (2.2). In particular, we do not require any monotonicity of the nonlinear term $f(x, u)$. All of those results improve the corresponding ones as in [12–14] essentially. To do this, let

$$g(x, u_h(x), v_h(x)) = \frac{1}{2}\left[f(x, u_h(x)) + f(x, v_h(x)) - B(u_h(x) - v_h(x))\right], \qquad x \in I_h.$$

It is readily proved that g has the following properties:

(i) $g(x, u_h(x), u_h(x)) = f(x, u_h(x))$; (2.6)

(ii) $g(x, u_h(x), v_h(x)) - g(x, v_h(x), u_h(x)) = B(v_h(x) - u_h(x))$; (2.7)

(iii) for any ordered pair of supersolution $\overline{u}_h$ and subsolution $\underline{u}_h$, and all $x \in I_h$,

$$\begin{cases} l_h \overline{u}_h(x) + P_h g(x, \overline{u}_h(x), \underline{u}_h(x)) \geq 0, \\ l_h \underline{u}_h(x) + P_h g(x, \underline{u}_h(x), \overline{u}_h(x)) \leq 0; \end{cases} \tag{2.8}$$

(iv) In the cone $\mathbf{K}(\underline{u}_h, \overline{u}_h)$, $g(x, u_h(x), v_h(x))$ is nonincreasing for u_h, and nondecreasing for v_h.

We now consider the following modified iteration

$$\begin{cases} l_h \overline{u}_h^{(k)}(x) = -P_h g(x, \overline{u}_h^{(k-1)}(x), \underline{u}_h^{(k-1)}(x)), & x \in I_h, \\ l_h \underline{u}_h^{(k)}(x) = -P_h g(x, \underline{u}_h^{(k-1)}(x), \overline{u}_h^{(k-1)}(x)), & x \in I_h, \\ \overline{u}_h^{(k)}(x) = \underline{u}_h^{(k)}(x) = 0, \qquad x = 0, 1. \end{cases} \tag{2.9}$$

For all $h \le h_1$, the above iteration is well defined.

Theorem 2.2. *Assume that $\{\underline{u}_h, \overline{u}_h\}$ is an ordered pair of supersolution and subsolution for (2.2), and $h \le \min(h_0, h_1)$. Then the sequences $\left\{\overline{u}_h^{(k)}(x)\right\}$ and $\left\{\underline{u}_h^{(k)}(x)\right\}$ defined by iteration (2.9) with the initial values $\overline{u}_h^{(0)}(x) = \overline{u}_h(x)$ and $\underline{u}_h^{(0)}(x) = \underline{u}_h(x)$ converge monotonically to the limits $\overline{u}_h^*(x)$ and $\underline{u}_h^*(x)$, respectively. Moreover for all $x \in \overline{I}_h$ and $k \ge 0$,*

$$\underline{u}_h(x) \le \underline{u}_h^{(k)}(x) \le \underline{u}_h^{(k+1)}(x) \le \underline{u}_h^*(x) \le \overline{u}_h^*(x) \le \overline{u}_h^{(k+1)}(x) \le \overline{u}_h^{(k)}(x) \le \overline{u}_h(x). \tag{2.10}$$

Besides, for any possible solution $u_h(x)$ of problem (2.2) in $\mathbf{K}(\underline{u}_h, \overline{u}_h)$, we have $u_h \in \mathbf{K}(\underline{u}_h^*, \overline{u}_h^*)$.

Proof. We use induction to assert that

$$\underline{u}_h^{(k)}(x) \le \underline{u}_h^{(k+1)}(x) \le \overline{u}_h^{(k+1)}(x) \le \overline{u}_h^{(k)}(x), \qquad x \in \overline{I}_h. \tag{2.11}$$

Firstly, by $(\mathbf{H_2})$ and (2.8),

$$\underline{u}_h^{(0)}(x) \le \underline{u}_h^{(1)}(x) \le \overline{u}_h^{(1)}(x) \le \overline{u}_h^{(0)}(x), \qquad x \in \overline{I}_h$$

which gives the desired result (2.11) for $k = 0$. Next assume that (2.11) is true. Then we have from $(\mathbf{H_2})$ and the property $(\mathbf{iv})$ of g that

$$\underline{u}_h^{(k+1)}(x) \le \underline{u}_h^{(k+2)}(x) \le \overline{u}_h^{(k+2)}(x) \le \overline{u}_h^{(k+1)}(x), \qquad x \in \overline{I}_h.$$

The induction is completed. Therefore there exist the limits $\overline{u}_h^*(x)$ and $\underline{u}_h^*(x)$ such that

$$\lim_{k\to\infty} \overline{u}_h^{(k)}(x) = \overline{u}_h^*(x), \quad \lim_{k\to\infty} \underline{u}_h^{(k)}(x) = \underline{u}_h^*(x)$$

and (2.10) holds.

Now, let $u_h(x)$ be any possible solution of problem (2.2) in $\mathbf{K}(\underline{u}_h, \overline{u}_h)$. Suppose that $u_h \in \mathbf{K}(\underline{u}_h^{(k)}, \overline{u}_h^{(k)})$ for some k. Subtracting the first equation of (2.2) from the first equation of (2.9), we get

$$l_h\left(\overline{u}_h^{(k+1)}(x) - u_h(x)\right) = P_h\left(g(x, u_h(x), u_h(x)) - g(x, \overline{u}_h^{(k)}(x), \underline{u}_h^{(k)}(x))\right) \ge 0, \qquad x \in I_h.$$

So by $(\mathbf{H_2})$, $u_h(x) \le \overline{u}_h^{(k+1)}(x)$. Similarly, $u_h(x) \ge \underline{u}_h^{(k+1)}(x)$. The above argument leads to that

$$\underline{u}_h^{(k)}(x) \le u_h(x) \le \overline{u}_h^{(k)}(x), \qquad x \in \overline{I}_h, \quad k = 0, 1, \cdots.$$

Letting $k \to \infty$, we see that $u_h \in \mathbf{K}(\underline{u}_h^*, \overline{u}_h^*)$. This completes the proof.

We obtain from (2.6) the following result immediately.

Theorem 2.3. *Assume that all the hypotheses in Theorem 2.2 hold, and let $\overline{u}_h^*(x)$ and $\underline{u}_h^*(x)$ be the limits obtained from the corresponding monotone sequences. If $\overline{u}_h^*(x) = \underline{u}_h^*(x)$ for all $x \in \overline{I}_h$, then $u_h(x) := \overline{u}_h^*(x) = \underline{u}_h^*(x)$ is the unique solution of problem (2.2) in $\mathbf{K}(\underline{u}_h, \overline{u}_h)$.*

3. A monotone approximation on uniform mesh

In this section, we investigate a fourth order approximation. Especially, we provide a sufficient condition for the monotone convergence of iteration (2.9). We focus on uniform mesh. Let $h = h_p$, $1 \le p \le N$, and

$$
\begin{aligned}
J_{i,p} &= \left(\int_{x_{p-1}}^{x_p} \frac{1}{a_i(t)} \mathrm{d}t \right)^{-1}, \\
E_{i,p} &= J_{i,p+1} \int_{x_p}^{x_{p+1}} \psi_{p,1}(x) \left(\int_x^{x_{p+1}} \frac{1}{a_i(t)} \mathrm{d}t \right) \mathrm{d}x + J_{i,p} \int_{x_{p-1}}^{x_p} \psi_{p,1}(x) \left(\int_{x_{p-1}}^{x} \frac{1}{a_i(t)} \mathrm{d}t \right) \mathrm{d}x, \\
F_{i,p} &= J_{i,p+1} \int_{x_p}^{x_{p+1}} \psi_{p,2}(x) \left(\int_x^{x_{p+1}} \frac{1}{a_i(t)} \mathrm{d}t \right) \mathrm{d}x + J_{i,p} \int_{x_{p-1}}^{x_p} \psi_{p,2}(x) \left(\int_{x_{p-1}}^{x} \frac{1}{a_i(t)} \mathrm{d}t \right) \mathrm{d}x, \\
G_{i,p} &= J_{i,p+1} \int_{x_p}^{x_{p+1}} \psi_{p,3}(x) \left(\int_x^{x_{p+1}} \frac{1}{a_i(t)} \mathrm{d}t \right) \mathrm{d}x + J_{i,p} \int_{x_{p-1}}^{x_p} \psi_{p,3}(x) \left(\int_{x_{p-1}}^{x} \frac{1}{a_i(t)} \mathrm{d}t \right) \mathrm{d}x
\end{aligned}
$$

where

$$
\psi_{p,1}(x) = -\frac{x - x_p}{2h} + \frac{(x - x_p)^2}{2h^2}, \;\; \psi_{p,2}(x) = 1 - \frac{(x - x_p)^2}{h^2}, \;\; \psi_{p,3}(x) = \frac{x - x_p}{2h} + \frac{(x - x_p)^2}{2h^2}.
$$

For any function $q_h(x) = (q_{h,1}(x), \cdots, q_{h,n}(x))^T$ on $\overline{I}_h$, we define

$$
l_h = \mathrm{diag}(l_{h,1}, \cdots, l_{h,n}), \qquad P_h = \mathrm{diag}(P_{h,1}, \cdots, P_{h,n})
$$

with

$$
\begin{aligned}
l_{h,i} q_{h,i}(x_p) &= -J_{i.p} q_{h,i}(x_{p-1}) + (J_{i,p} + J_{i,p+1}) q_{h,i}(x_p) - J_{i,p+1} q_{h,i}(x_{p+1}), \\
P_{h,i} q_{h,i}(x_p) &= E_{i,p} q_{h,i}(x_{p-1}) + F_{i,p} q_{h,i}(x_p) + G_{i,p} q_{h,i}(x_{p+1}), \\
&\qquad\qquad 1 \le p \le N-1, \qquad 1 \le i \le n.
\end{aligned}
$$

Then we obtain the following fourth order approximation to (2.1) (see [14]),

$$
\begin{cases} l_h u_h(x) + P_h f(x, u_h(x)) = 0, & x \in I_h, \\ u_h(0) = u_h(1) = 0. \end{cases} \tag{3.1}
$$

Let $J_{h,i} = (J_{p,q}^{(i)})$ be the tridiagonal matrix with the elements

$$
J_{p,p-1}^{(i)} = -J_{i,p}, \;\; J_{p,p}^{(i)} = J_{i,p} + J_{i,p+1}, \;\; J_{p,p+1}^{(i)} = -J_{i,p+1}, \qquad 1 \le p \le N-1, \; 1 \le i \le n,
$$

and let $D_{h,i} = (D_{p,q}^{(i)})$ denote the tridiagonal matrix with the elements

$$
D_{p,p-1}^{(i)} = E_{i,p}, \;\; D_{p,p}^{(i)} = F_{i,p}, \;\; D_{p,p+1}^{(i)} = G_{i,p}, \qquad 1 \le p \le N-1, \; 1 \le i \le n.
$$

Moreover let

$$
\begin{aligned}
&U_{h,i} = (u_{h,i}(x_1), \cdots, u_{h,i}(x_{N-1}))^T,\\
&F_i(u_h) = (f_i(x_1, u_h(x_1)), \cdots, f_i(x_{N-1}, u_h(x_{N-1})))^T,\\
&Q_{h,i}(u_h) = (-J_{i,1}u_{h,i}(0), 0, \cdots, 0, -J_{i,N}u_{h,i}(1))^T,\\
&H_{h,i}(u_h) = (E_{i,1}f_i(0, u_h(0)), 0, \cdots, 0, G_{i,N-1}f_i(1, u_h(1)))^T.
\end{aligned}
$$

Then the approximation (3.1) may be described by

$$
\begin{cases} J_{h,i}U_{h,i} + D_{h,i}F_i(u_h) + Q_{h,i}(u_h) + H_{h,i}(u_h) = 0, & 1 \le i \le n,\\ u_h(0) = u_h(1) = 0. \end{cases} \tag{3.2}
$$

We shall show the monotonicity of scheme (3.1). We say that a vector Z (or a matrix $A = (A_{p,q})$) is nonnegative, denoted by $Z \ge 0$ (or $A \ge 0$), if all of its elements are nonnegative.

Lemma 3.1 (see [14]). *If $\alpha_1 \le \sqrt{3}\alpha_0$, then $D_{h,i} \ge 0$.*

The following lemma tells us that for sufficiently small h, we can delete the condition in the previous lemma.

Lemma 3.2 *(see [14]). There exists a positive constant h^* such that for all $h \le h^*$, $D_{h,i} \ge 0$.*

Following the work of [14] we can estimate the value of h^*. But in practical problems, we can calculate directly the values of $E_{i,p}$ and $G_{i,p}$ to obtain more precise value of h^*.

We next introduce the concept of monotone matrix. A matrix $A = (A_{p,q})$ is called a monotone matrix, if $AZ \ge 0$ implies $Z \ge 0$ for any real vector Z (see [8, 16]). In particular, a matrix $A = (A_{p,q})$ is of positive–type, if it fulfills the following conditions:

(i) $A_{p,p} > 0$ and for $p \ne q$, $A_{p,q} \le 0$;

(ii) $d_p = \dfrac{-\sum_{q\ne p} A_{p,q}}{A_{p,p}} \le 1$ and the set $\mathcal{N}(A) = \{p|\ d_p < 1\}$ is not empty;

(iii) for any $p_1 \notin \mathcal{N}(A)$, there exists $p_2 \in \mathcal{N}(A)$ such that

$$A_{p_1,q_1}A_{q_1,q_2}\cdots A_{q_j,p_2} \ne 0.$$

Any matrix of positive–type is monotone (see [8]).

Lemma 3.3. *For all h and i, $J_{h,i}$ is monotone.*

Proof. We can check that $J_{h,i}$ is a matrix of positive-type.

Now let h^* be the constant occurring in Lemma 3.2, and

$$
\overline{h} = \begin{cases} \text{arbitrary positive constant}, & \alpha_1 \le \sqrt{3}\alpha_0,\\ h^*, & \alpha_1 > \sqrt{3}\alpha_0. \end{cases}
$$

The combination of Lemma 3.1–Lemma 3.3 leads to the following result.

Theorem 3.1. *The approximation (3.1) is monotone with $h_0 = \overline{h}$ and arbitrary positive constant h_1.*

Proof. Lemma 3.1 and Lemma 3.2 imply the positivity of P_h for all $h \le h_0$. Hence it suffices to show that (3.1) satisfies condition ($\mathbf{H_2}$), that is , the system

$$\begin{cases} l_h u_h(x) \ge 0, & x \in I_h, \\ u_h(0) \ge 0, \quad u_h(1) \ge 0 \end{cases} \tag{3.3}$$

implies that $u_h(x) \ge 0$ for all $x \in \overline{I}_h$ and $h \ge 0$. In terms of matrices, (3.3) may be described by

$$\begin{cases} J_{h,i} U_{h,i} \ge -Q_{h,i}(u_h), & 1 \le i \le n, \\ u_h(0) \ge 0, \quad u_h(1) \ge 0 \end{cases} \tag{3.4}$$

where $J_{h,i}$, $Q_{h,i}(u_h)$, and $U_{h,i}$ are the same matrices and vectors as in (3.2). By the monotonicity of $J_{h,i}$ and the nonnegativities of $-Q_{h,i}(u_h)$, we use (3.4) to conclude that $U_{h,i} \ge 0$ and so $u_h(x) \ge 0$ for all $x \in \overline{I}_h$ and $h \ge 0$. This implies condition ($\mathbf{H_2}$).

By Theorem 3.1, all results in previous section are valid for (3.1). We now provide a condition ensuring that the iteration (2.9) converges monotonically to the unique solution of (3.1). For this purpose, we define the following discrete norms:

$$\|z_h\|^2 = \max_{1 \le i \le n} \sum_{p=0}^{N} h z_{h,i}^2(x_p), \qquad |z_h|_1^2 = \max_{1 \le i \le n} \sum_{p=1}^{N} \frac{(z_{h,i}(x_p) - z_{h,i}(x_{p-1}))^2}{h}.$$

Lemma 3.4 (see [14]). *If $z_{h,i}(0) = z_{h,i}(1) = 0$, and $y_i(x_p) \ge \dfrac{\alpha_0}{h}$ for $1 \le p \le N$, then*

$$\begin{aligned} \sum_{p=1}^{N-1} &(-y_i(x_p) z_{h,i}(x_{p-1}) + (y_i(x_p) + y_i(x_{p+1})) z_{h,i}(x_p) - y_i(x_{p+1})) z_{h,i}(x_p) \\ &= \sum_{p=1}^{N} y_i(x_p)(z_{h,i}(x_p) - z_{h,i}(x_{p-1}))^2 \\ &\ge \alpha_0 \sum_{p=1}^{N} \frac{1}{h}(z_{h,i}(x_p) - z_{h,i}(x_{p-1}))^2. \end{aligned}$$

Lemma 3.5 *(see [14]). If $z_h(0) = 0$ or $z_h(1) = 0$, then $\|z_h\| \le |z_h|_1$.*

Lemma 3.6 *(Theorem 3, page 298 of [17]). Let $\mathcal{V}$ denote the identity matrix. If a matrix $A = \mathcal{V} - S$, $S \ge 0$ and for certain norm $\|\cdot\|$, $\|S\| < 1$, then A is monotone.*

Let

$$\alpha_4 = \min\left\{\frac{3\alpha_1^2 - \alpha_0^2}{24\alpha_0\alpha_1}, \frac{5}{12}\right\}, \quad \alpha_5 = \min\left\{\frac{5\alpha_1}{6\alpha_0}, 2\right\}.$$

Theorem 3.2. *Assume that $\{\overline{u}_h, \underline{u}_h\}$ is an ordered pair of supersolution and subsolution for (3.1). Let $h \le \overline{h}$ and $\overline{M} = \max_{i,j} B_{i,j}$. If*

$$n\overline{M} < \max\left\{\frac{\alpha_0}{2\alpha_4 + \alpha_5}, \alpha_0, \frac{8{\alpha_0}^3}{{\alpha_1}^2}\right\},$$

then the iteration (2.9) with the initial values $\overline{u}_h^{(0)}(x) = \overline{u}_h(x)$ *and* $\underline{u}_h^{(0)}(x) = \underline{u}_h(x)$ *yields the sequences* $\{\overline{u}_h^{(k)}(x)\}$ *and* $\{\underline{u}_h^{(k)}(x)\}$ *converging to the unique solution* $u_h(x)$ *of (3.1) in* $\mathbf{K}(\underline{u}_h, \overline{u}_h)$. *Moreover,*

$$\underline{u}_h(x) \le \underline{u}_h^{(k)}(x) \le \underline{u}_h^{(k+1)}(x) \le u_h(x) \le \overline{u}_h^{(k+1)}(x) \le \overline{u}_h^{(k)}(x) \le \overline{u}_h(x), \qquad x \in \overline{I}_h. \quad (3.5)$$

Proof. By Theorem 2.2, there exist limits $\overline{u}_h^*(x)$ and $\underline{u}_h^*(x)$ such that

$$\lim_{k\to\infty} \overline{u}_h^k(x) = \overline{u}_h^*(x), \qquad \lim_{k\to\infty} \underline{u}_h^k(x) = \underline{u}_h^*(x)$$

and for $x \in \overline{I}_h$,

$$\underline{u}_h(x) \le \underline{u}_h^{(k)}(x) \le \underline{u}_h^{(k+1)}(x) \le \underline{u}_h^*(x) \le \overline{u}_h^*(x) \le \overline{u}_h^{(k+1)}(x) \le \overline{u}_h^{(k)}(x) \le \overline{u}_h(x).$$

So by Theorem 2.3, it suffices to check that for all $x \in \overline{I}_h$, $\overline{u}_h^*(x) = \underline{u}_h^*(x)$. Let $w_h^*(x) = \overline{u}_h^*(x) - \underline{u}_h^*(x)$. Then $w_h^*(x) \ge 0$. By (2.9), for all $x \in I_h$,

$$\begin{cases} l_h \overline{u}_h^*(x) = -P_h g(x, \overline{u}_h^*(x), \underline{u}_h^*(x)), \\ l_h \underline{u}_h^*(x) = -P_h g(x, \underline{u}_h^*(x), \overline{u}_h^*(x)). \end{cases}$$

Furthermore, by (2.7),

$$\begin{cases} l_h w_h^*(x) = P_h \left(B w_h^*(x)\right), \qquad x \in I_h, \\ w_h^*(0) = w_h^*(1) = 0 \end{cases}$$

or equivalently,

$$\begin{cases} l_{h,i} w_{h,i}^*(x_p) = P_{h,i}\left(\sum_{j=1}^n B_{i,j} w_{h,j}^*(x_p)\right), \qquad 1 \le p \le N-1,\ 1 \le i \le n, \\ w_h^*(0) = w_h^*(1) = 0. \end{cases} \quad (3.6)$$

We now consider two different cases. In the first case, $n\overline{M} < \dfrac{\alpha_0}{2\alpha_4 + \alpha_5}$. By multiplying the first equation of (3.6) by $w_{h,i}^*(x_p)$ and summing the result over all x_p, we obtain from Lemma 3.4 that

$$\alpha_0 \sum_{p=1}^N \frac{1}{h}\left(w_{h,i}^*(x_p) - w_{h,i}^*(x_{p-1})\right)^2 \le \sum_{p=1}^{N-1} P_{h,i}\left(\sum_{j=1}^n B_{i,j} w_{h,j}^*(x_p)\right) w_{h,i}^*(x_p). \quad (3.7)$$

By Lemma 3.1 and Lemma 3.2, it is easy to show that

$$0 \le E_{i,p} \le J_{i,p} \int_{x_{p-1}}^{x_p} \frac{1}{a_i(t)} \left(\frac{(t-x_p)^2}{4h} - \frac{(t-x_p)^3}{6h^2}\right) \mathrm{d}t \le \frac{5}{12}h.$$

Similarly,

$$0 \le F_{i,p} \le 2h, \qquad 0 \le G_{i,p} \le \frac{5}{12}h.$$

On the other hand, we have

$$0 \le E_{i,p} \le -\frac{J_{i,p+1}}{24\alpha_1}h^2 + \frac{J_{i,p}}{8\alpha_0}h^2 \le \frac{3\alpha_1^2 - \alpha_0^2}{24\alpha_0\alpha_1}h.$$

Similarly,

$$0 \le F_{i,p} \le \frac{5\alpha_1}{6\alpha_0}h, \qquad 0 \le G_{i,p} \le \frac{3\alpha_1^2 - \alpha_0^2}{24\alpha_0\alpha_1}h.$$

Therefore

$$0 \le E_{i,p} \le \alpha_4 h, \qquad 0 \le F_{i,p} \le \alpha_5 h, \qquad 0 \le G_{i,p} \le \alpha_4 h.$$

Accordingly,

$$\begin{aligned}
&\sum_{p=1}^{N-1} P_{h,i}\left(\sum_{j=1}^{n} B_{i,j} w_{h,j}^*(x_p)\right) w_{h,i}^*(x_p) \\
&\quad\le \overline{M}\sum_{j=1}^{n}\left(\sum_{p=1}^{N-1}\alpha_4 h w_{h,j}^*(x_{p-1}) w_{h,i}^*(x_p) + \sum_{p=1}^{N-1}\alpha_5 h w_{h,j}^*(x_p) w_{h,i}^*(x_p)\right. \\
&\qquad\qquad\qquad \left. + \sum_{p=1}^{N-1}\alpha_4 h w_{h,j}^*(x_{p+1}) w_{h,i}^*(x_p)\right) \\
&\quad\le (2\alpha_4 + \alpha_5) n\overline{M}\|w_h^*(x)\|^2.
\end{aligned}$$

Hence by (3.7) and Lemma 3.5,

$$\alpha_0 |w_h^*(x)|_1^2 \le (2\alpha_4 + \alpha_5) n\overline{M} |w_h^*(x)|_1^2.$$

This fact and the boundary conditions lead to the conclusion. In the second case, $n\overline{M} < \max\left\{\alpha_0, \dfrac{8{\alpha_0}^3}{\alpha_1^2}\right\}$. Set

$$\begin{aligned}
W_{h,i}^* &= \left(w_{h,i}^*(x_1), \cdots, w_{h,i}^*(x_{N-1})\right)^T, \\
\overline{W}_{h,i}^* &= \left(\sum_{j=1}^{n} B_{i,j} w_{h,j}^*(x_1), \cdots, \sum_{j=1}^{n} B_{i,j} w_{h,j}^*(x_{N-1})\right)^T, \\
\overline{W}_h^* &= \left(\sum_{j=1}^{n} w_{h,j}^*(x_1), \cdots, \sum_{j=1}^{n} w_{h,j}^*(x_{N-1})\right)^T.
\end{aligned}$$

Then (3.6) reads

$$\begin{cases} W_{h,i}^* = J_{h,i}^{-1} D_{h,i} \overline{W}_{h,i}^*, & 1 \le i \le n, \\ w_h^*(0) = w_h^*(1) = 0. \end{cases}$$

Therefore

$$\begin{cases} \left(\mathcal{V} - \overline{M}\displaystyle\sum_{i=1}^{n} J_{h,i}^{-1} D_{h,i}\right)\overline{W}_h^* \le 0, \\ w_h^*(0) = w_h^*(1) = 0. \end{cases}$$

If the matrix $\mathcal{V}-\overline{M}\sum_{i=1}^{n} J_{h,i}^{-1}D_{h,i}$ is monotone, then $\overline{W}_h^* \le 0$ and so by the nonnegativity of $\overline{W}_h^*$, $\overline{W}_h^* = 0$. By Lemma 3.6, it suffices to prove that for certain norm $\|\cdot\|$,

$$\|\overline{M}\sum_{i=1}^{n} J_{h,i}^{-1}D_{h,i}\| < 1. \tag{3.8}$$

To do this, let $J_{h,i}^{-1} = \left(J_{p,q}^{(i)^{-1}}\right)$. By using the usual method for inverting a symmetric tridiagonal matrix, we obtain

$$J_{p,q}^{(i)^{-1}} = \begin{cases} \sum_{l=1}^{p}\frac{1}{J_{i,l}}\sum_{l=q+1}^{N}\frac{1}{J_{i,l}}\Big/\sum_{l=1}^{N}\frac{1}{J_{i,l}}, & p \le q, \\ \sum_{l=1}^{q}\frac{1}{J_{i,l}}\sum_{l=p+1}^{N}\frac{1}{J_{i,l}}\Big/\sum_{l=1}^{N}\frac{1}{J_{i,l}}, & p > q. \end{cases}$$

In particular,

$$\Delta_{i,p} = \sum_{q=1}^{N-1}|J_{p,q}^{(i)^{-1}}| = \left(\sum_{l=1}^{p}\frac{1}{J_{i,l}}\left(\sum_{q=p}^{N-1}\sum_{l=q+1}^{N}\frac{1}{J_{i,l}}\right) + \sum_{l=p+1}^{N}\frac{1}{J_{i,l}}\left(\sum_{q=1}^{p-1}\sum_{l=1}^{q}\frac{1}{J_{i,l}}\right)\right)\Big/\sum_{l=1}^{N}\frac{1}{J_{i,l}}.$$

Furthermore,

$$\begin{aligned} \Delta_{i,p} &\le \frac{h}{2\alpha_0}\left((N-p)(N+1-p)\sum_{l=1}^{p}\frac{1}{J_{i,l}} + p(p-1)\sum_{l=p+1}^{N}\frac{1}{J_{i,l}}\right)\Big/\sum_{l=1}^{N}\frac{1}{J_{i,l}} \\ &\le \frac{h}{2\alpha_0}((N-p)(N+1-p)+p(p-1)) \\ &\le \frac{h}{2\alpha_0}(N^2-N) = \frac{h}{2\alpha_0}(\frac{1}{h}-1) \end{aligned}$$

whence $\|J_{h,i}^{-1}\|_\infty = \max_p \Delta_{i,p} \le \frac{1}{2\alpha_0}(\frac{1}{h}-1)$. On the other hand,

$$E_{i,p}+F_{i,p}+G_{i,p} = J_{i,p+1}\int_{x_p}^{x_{p+1}}\frac{t-x_p}{a_i(t)}\mathrm{d}t + J_{i,p}\int_{x_{p-1}}^{x_p}\frac{x_p-t}{a_i(t)}\mathrm{d}t \le 2h$$

whence $\|D_{h,i}\|_\infty \le 2h$. So we have

$$\|\overline{M}\sum_{i=1}^{n}J_{h,i}^{-1}D_{h,i}\|_\infty \le \overline{M}\sum_{i=1}^{n}\|J_{h,i}^{-1}\|_\infty\|D_{h,i}\|_\infty \le \frac{n\overline{M}}{\alpha_0}(1-h).$$

Also we have

$$\begin{aligned} \Delta_{i,p} &\le \frac{h}{2\alpha_0}\left((N-p)(N+1-p)\sum_{l=1}^{p}\frac{1}{J_{i,l}} + p(p-1)\sum_{l=p+1}^{N}\frac{1}{J_{i,l}}\right)\Big/\sum_{l=1}^{N}\frac{1}{J_{i,l}} \\ &\le \frac{\alpha_1 h}{2\alpha_0^2}((N-p)(N+1-p)p + p(p-1)(N-p)) \\ &= \frac{\alpha_1 h}{2\alpha_0^2}p(N-p) \le \frac{\alpha_1 h N^2}{8\alpha_1^2} = \frac{\alpha_1}{8\alpha_0^2 h} \end{aligned}$$

and

$$E_{i,p}+F_{i,p}+G_{i,p}=J_{i,p+1}\int_{x_p}^{x_{p+1}}\frac{t-x_p}{a_i(t)}\mathrm{d}t+J_{i,p}\int_{x_{p-1}}^{x_p}\frac{x_p-t}{a_i(t)}\mathrm{d}t\le\frac{\alpha_0}{\alpha_1}h.$$

Hence $\|J_{h,i}^{-1}\|_\infty\le\dfrac{\alpha_1}{8\alpha_0^2h}$, $\|D_{h,i}\|_\infty\le\dfrac{\alpha_1}{\alpha_0}h$, and

$$\|\overline{M}\sum_{i=1}^n J_{h,i}^{-1}D_{h,i}\|_\infty\le\overline{M}\sum_{i=1}^n\|J_{h,i}^{-1}\|_\infty\|D_{h,i}\|_\infty\le\frac{n\overline{M}\alpha_1^2}{8\alpha_0^3}.$$

So (3.8) holds with norm $\|\cdot\|_\infty$ as long as

$$\min\left\{\frac{n\overline{M}}{\alpha_0}(1-h),\frac{n\overline{M}\alpha_1^2}{8\alpha_0^3}\right\}<1.$$

This completes the proof.

We now estimate the error between $\overline{u}_h^{(k)}(x)$ and $u_h(x)$, and the error between $\underline{u}_h^{(k)}(x)$ and $u_h(x)$.

Theorem 3.3. *Assume that the hypotheses in Theorem 3.2 hold, and* $(4\alpha_4+2\alpha_5)n\overline{M}<\alpha_0$. *Then*

$$|\overline{u}_h^{(k)}(x)-u_h(x)|_1^2+|\underline{u}_h^{(k)}(x)-u_h(x)|_1^2\le\gamma^k\left(\|\overline{u}_h^{(0)}(x)-u_h(x)\|^2+\|\underline{u}_h^{(0)}(x)-u_h(x)\|^2\right)$$

with

$$\gamma=\frac{(2\alpha_4+\alpha_5)n\overline{M}}{\alpha_0-(2\alpha_4+\alpha_5)n\overline{M}}<1.$$

Proof. Let $\overline{z}_h^{(k)}(x)=\overline{u}_h^{(k)}(x)-u_h(x)$ and $\underline{z}_h^{(k)}(x)=\underline{u}_h^{(k)}(x)-u_h(x)$. Then $\overline{z}_h^{(k)}(x)\ge0$ and $\underline{z}_h^{(k)}(x)\le0$ for all $x\in\overline{I}_h$. Moreover,

$$\begin{cases} l_{h,i}\overline{z}_{h,i}^{(k+1)}(x_p)=P_{h,i}\left(f_i(x,u_h(x))-g_i(x,\overline{u}_h^{(k)}(x_p),\underline{u}_h^{(k)}(x_p))\right), \\ \overline{z}_{h,i}^{(k+1)}(0)=\overline{z}_{h,i}^{(k+1)}(1)=0, \qquad 1\le p\le N-1,\ 1\le i\le n. \end{cases}\tag{3.9}$$

By the definition of g and the property (**i**) in Definition 2.1, we know that

$$f_i(x,u_h(x))-g_i(x,\overline{u}_h^{(k)}(x_p),\underline{u}_h^{(k)}(x_p))\le\sum_{j=1}^n B_{i,j}\left(\overline{z}_{h,j}^{(k)}(x_p)-\underline{z}_{h,j}^{(k)}(x_p)\right).$$

Multiplying the first equation of (3.9) by $\overline{z}_{h,i}^{(k+1)}(x_p)$ and summing the result over all x_p, we obtain from Lemma 3.4 that

$$\begin{aligned}\alpha_0\sum_{p=1}^N\frac{1}{h}&\left(\overline{z}_{h,i}^{(k+1)}(x_p)-\overline{z}_{h,i}^{(k+1)}(x_{p-1})\right)^2\\ &\le\sum_{p=1}^{N-1}P_{h,i}\left(\sum_{j=1}^n B_{i,j}(\overline{z}_{h,j}^{(k)}(x_p)-\underline{z}_{h,j}^{(k)}(x_p))\right)\overline{z}_{h,i}^{(k+1)}(x_p)\\ &\le(\alpha_4+\frac{1}{2}\alpha_5)n\overline{M}\left(\|\overline{z}_h^{(k)}(x)\|^2+2\|\overline{z}_h^{(k+1)}(x)\|^2+\|\underline{z}_h^{(k)}(x)\|^2\right).\end{aligned}$$

A similar estimate is valid for $\underline{z}_h^{(k)}(x)$. The combination yields that

$$\begin{aligned}&\alpha_0\left(|\overline{z}_h^{(k+1)}(x)|_1^2+|\underline{z}_h^{(k+1)}(x)|_1^2\right)\\&\qquad\le(2\alpha_4+\alpha_5)n\overline{M}\left(\|\overline{z}_h^{(k)}(x)\|^2+\|\underline{z}_h^{(k)}(x)\|^2+\|\overline{z}_h^{(k+1)}(x)\|^2+\|\underline{z}_h^{(k+1)}(x)\|^2\right).\end{aligned}$$

By virtue of Lemma 3.5,

$$\begin{aligned}|\overline{z}_h^{(k)}(x)|_1^2+|\underline{z}_h^{(k)}(x)|_1^2&\le\gamma^{k-1}\left(|\overline{z}_h^{(1)}(x)|_1^2+|\underline{z}_h^{(1)}(x)|_1^2\right),\\|\overline{z}_h^{(1)}(x)|_1^2+|\underline{z}_h^{(1)}(x)|_1^2&\le\gamma\left(\|\overline{z}_h^{(0)}(x)\|^2+\|\underline{z}_h^{(0)}(x)\|^2\right)\end{aligned}$$

and the conclusion follows.

Theorem 3.3 shows the geometric convergence rate of the iteration (2.9).

4. Numerical Results

This section is devoted to numerical results. We consider the following system

$$\begin{cases}-u_1''(x)+f_1(x,u_1,u_2)=0, & 0<x<1,\\ -u_2''(x)+f_2(x,u_1,u_2)=0, & 0<x<1,\\ u_1(x)=u_2(x)=0, & x=0,1\end{cases}\tag{4.1}$$

where

$$\begin{aligned}f_1(x,u_1,u_2)&=-p_1(x)\cos(q_1(x)u_2(x)),\\f_2(x,u_1,u_2)&=-p_2(x)\cos(q_2(x)u_1(x)).\end{aligned}$$

The functions $p_i(x),q_i(x)\in C^0(I)$ and $|p_i(x)|,|q_i(x)|\le\alpha$ for $x\in I$. We solve (4.1) by approximation (3.1). In this case, (3.1) is reduced to

$$\begin{cases}-u_{h,1}(x_{p-1})+2u_{h,1}(x_p)-u_{h,1}(x_{p+1})+\dfrac{h^2}{12}\left(f_1(x_{p-1},u_{h,1}(x_{p-1}),u_{h,2}(x_{p-1}))\right.\\ \qquad\left.+\,10f_1(x_p,u_{h,1}(x_p),u_{h,2}(x_p))+f_1(x_{p+1},u_{h,1}(x_{p+1}),u_{h,2}(x_{p+1}))\right)=0,\\ -u_{h,2}(x_{p-1})+2u_{h,2}(x_p)-u_{h,2}(x_{p+1})+\dfrac{h^2}{12}\left(f_2(x_{p-1},u_{h,1}(x_{p-1}),u_{h,2}(x_{p-1}))\right.\\ \qquad\left.+\,10f_2(x_p,u_{h,1}(x_p),u_{h,2}(x_p))+f_2(x_{p+1},u_{h,1}(x_{p+1}),u_{h,2}(x_{p+1}))\right)=0,\\ u_{h,1}(0)=u_{h,1}(1)=u_{h,2}(0)=u_{h,2}(1)=0.\end{cases}\tag{4.2}$$

Clearly (4.2) is monotone for all h. Since the functions $p_i(x)$ and $q_i(x)$ may oscillate arbitrarily, the monotonicity of the functions f is destroyed usually. Now let $\alpha>0$,

$$\overline{u}_{h,i}(x)=-\underline{u}_{h,i}(x)=\alpha x(1-x),\qquad x\in\overline{I}_h,\quad i=1,2$$

and

$$B=\begin{pmatrix}0&\alpha^2\\\alpha^2&0\end{pmatrix}.$$

It can be verified that $\{\overline{u}_h, \underline{u}_h\}$ is an ordered pair of supersolution and subsolution.

We first take $\alpha = 1/3$ and $p_i(x) \equiv q_i(x) \equiv 1/3$, $i = 1, 2$. We use the iteration (2.9) to solve (4.2). In this case, (2.9) is reduced to

$$\begin{cases} -\overline{u}_h^{(k+1)}(x_{p-1}) + 2\overline{u}_h^{(k+1)}(x_p) - \overline{u}_h^{(k+1)}(x_{p+1}) = -\dfrac{h^2}{12}\Big(g(x_{p-1}, \overline{u}_h^{(k)}(x_{p-1}), \underline{u}_h^{(k)}(x_{p-1})) \\ \qquad\qquad + 10g(x_p, \overline{u}_h^{(k)}(x_p), \underline{u}_h^{(k)}(x_p)) + g(x_{p+1}, \overline{u}_h^{(k)}(x_{p+1}), \underline{u}_h^{(k)}(x_{p+1}))\Big), \\ -\underline{u}_h^{(k+1)}(x_{p-1}) + 2\underline{u}_h^{(k+1)}(x_p) - \underline{u}_h^{(k+1)}(x_{p+1}) = -\dfrac{h^2}{12}\Big(g(x_{p-1}, \underline{u}_h^{(k)}(x_{p-1}), \overline{u}_h^{(k)}(x_{p-1})) \\ \qquad\qquad + 10g(x_p, \underline{u}_h^{(k)}(x_p), \overline{u}_h^{(k)}(x_p)) + g(x_{p+1}, \underline{u}_h^{(k)}(x_{p+1}), \overline{u}_h^{(k)}(x_{p+1}))\Big), \\ \overline{u}_h^{(k+1)}(x) = \underline{u}_h^{(k+1)}(x) = 0, \qquad x = 0, 1 \end{cases}$$

with the initial values $\overline{u}_{h,i}^{(0)}(x_p) = \overline{u}_{h,i}(x_p)$ and $\underline{u}_{h,i}^{(0)}(x_p) = \underline{u}_{h,i}(x_p)$. Numerical results in Fig. 4.1 and Fig. 4.2 (h=1/20) show that the sequence $\left\{\overline{u}_h^{(k)}(x)\right\}$ is nonincreasing and $\left\{\underline{u}_h^{(k)}(x)\right\}$ is nondecreasing. It agrees the monotonicity described in Theorem 2.2 or Theorem 3.2. Furthermore, by Theorem 3.2, both of them converge to the unique solution of (4.2) in $\mathbf{K}(\underline{u}_h, \overline{u}_h)$. In actual calculation, if

$$\max_i \max_p |\overline{u}_{h,i}^{(k+1)}(x_p) - \overline{u}_{h,i}^{(k)}(x_p)| < 10^{-5}, \tag{4.3}$$

then we take $\overline{u}_h^{(k+1)}(x)$ as the approximate solution of (4.1). Numerical results are listed in Table 4.1. Similarly, if (4.3) holds for $\underline{u}_{h,i}^{(k+1)}(x_p)$ and $\underline{u}_{h,i}^{(k)}(x_p)$, then we take $\underline{u}_h^{(k+1)}(x)$ as the approximate solution of (4.1). The corresponding results are given in Table 4.2. Since the results are symmetric with respect to the central point, we only list the half results. Table 4.1 and Table 4.2 support the theoretical analysis in Theorem 3.2.

Next, we take $\alpha = 2$ and $p_i(x) \equiv q_i(x) \equiv 2$, $i = 1, 2$. In this case, we get the same results as in the first example, for instance, the monotonicity described in Theorem 2.2. In addition, we find that the sequence $\left\{\overline{u}_h^{(k)}(x)\right\}$ and $\left\{\underline{u}_h^{(k)}(x)\right\}$ have the same limit and so it is the unique solution of the resulting problem in $\mathbf{K}(\underline{u}_h, \overline{u}_h)$. Whereas the condition of Theorem 3.2 is now destroyed. Thus the condition in Theorem 3.2 is only a sufficient condition.

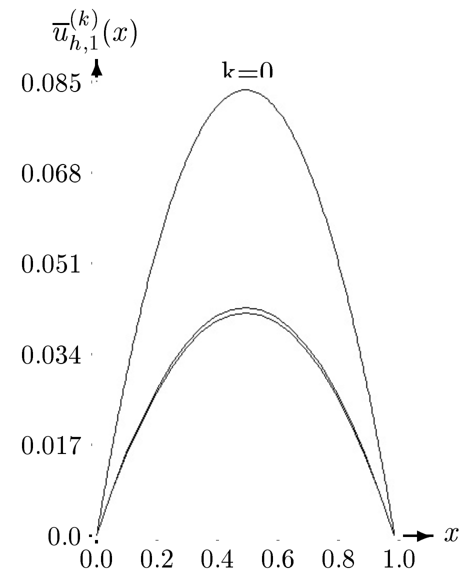
$\overline{u}_{h,1}^{(k)}(x)$
$\overline{u}_{h,2}^{(k)}(x)$
k=0
0.085
0.068
0.051
0.034
0.017
0.0
0.0 0.2 0.4 0.6 0.8 1.0
x

Fig. 4.1

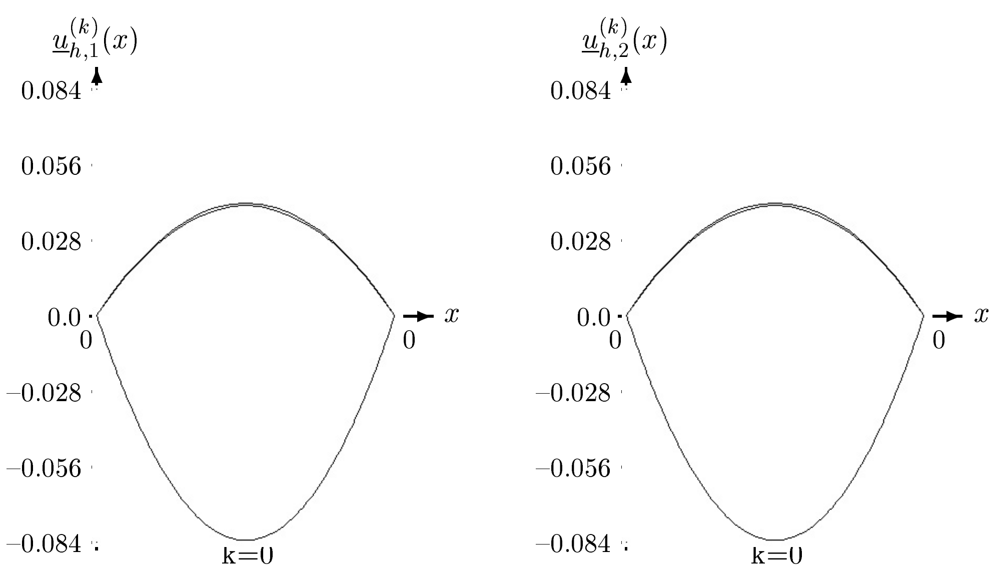
$\underline{u}_{h,1}^{(k)}(x)$
$\underline{u}_{h,2}^{(k)}(x)$
0.084
0.056
0.028
0.0
0
x
–0.028
–0.056
–0.084
k=0

Fig. 4.2

Table 4.1

	$N = 10$		$N = 30$	
x_p	$u_1(x_p)$	$u_2(x_p)$	$u_1(x_p)$	$u_2(x_p)$
0.1	0.014999	0.014999	0.014999	0.014999
0.2	0.026665	0.026665	0.026665	0.026665
0.3	0.034998	0.034998	0.034998	0.034998
0.4	0.039997	0.039997	0.039997	0.039997
0.5	0.041664	0.041664	0.041664	0.041664

Table 4.2

	$N = 10$		$N = 30$	
x_p	$u_1(x_p)$	$u_2(x_p)$	$u_1(x_p)$	$u_2(x_p)$
0.1	0.014999	0.014999	0.014999	0.014999
0.2	0.026665	0.026665	0.026665	0.026665
0.3	0.034998	0.034998	0.034998	0.034998
0.4	0.039997	0.039997	0.039997	0.039997
0.5	0.041664	0.041664	0.041664	0.041664

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