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CASCADIC MULTIGRID FOR PARABOLIC PROBLEMS*[1)]

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Abstract

In this paper, we develop the cascadic multigrid method for parabolic problems. The optimal convergence accuracy and computation complexity are obtained.

Key words: Cascadic multigrid, Finite element, Parabolic problem.

1. Introduction

Bornemann and Deuflhard [2][3] have presented a new type of multigrid methods, the so-called cascadic multigrid. Compared with usual multigrid methods, it requires no coarse grid corrections at all that may be viewed as a "one way" multigrid. Another distinctive feature is performing more iterations on coarser levels so as to obtain less iterations on finer levels. Numerical experiments show that this method is very effective for second order elliptic problems.

In the paper, we will consider the cascadic multigrid for parabolic problems. Here we must treat the effect of the time discrete step. As pointed out in [1], for a small time step $\tau \leq O(h^2)$, where h is the space mesh size, some standard iterative methods, like the Richardson iteration can guarantee a good convergence for the discrete system. But for a relative large time step τ, [1] recommended multigrid methods, see [4] for details. Now we consider to use the cascadic multigrid. Similar to the second order elliptic problem, it is proved that the cascadic multigrid with CG iteration as a smoother is accurate with the optimal complexity in 3D and 2D, and nearly optimal in 1D. As for other traditional iterative methods, like the Richardson iteration, the cascadic multigrid still yields the optimal accuracy and complexity in 3D, 2D and in a certain case of 1D. Notice that for the second order elliptic problem, the cascadic multigrid with these iterative methods gives the optimal accuracy and computation complexity only in 3D and nearly optimal in 2D. They cannot be used for 1D.

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2. Model Problem and Its Finite Elment Approximation

Consider the following parabolic problem: to find $u(x,t)$ such that

$$\begin{cases} \dfrac{\partial u}{\partial t}+\mathcal{L}u &= f \quad in \quad \Omega\times[0,T], \\ u(x,t) &= 0 \quad in \quad \partial\Omega\times[0,T], \\ u(x,0) &= u_0(x), \end{cases} \tag{2.1}$$

where $\Omega\subset R^d$ $(d=1,2,3)$ is a bounded domain, $f\in L^2(\Omega)$. $\mathcal{L}$ is an elliptic operator

$$\mathcal{L}u=-\sum_{i,j=1}^{d}\frac{\partial}{\partial x_i}(a_{ij}(x)\frac{\partial u}{\partial x_j}). \tag{2.2}$$

Here $a_{ij}(x)$ satisfies

$$c\xi^t\xi\le\sum_{i,j=1}^{d}a_{ij}\xi_i\xi_j\le C\xi^t\xi \quad \forall x\in\Omega,\xi\in R^d, \tag{2.3}$$

where c,C are positive constants.

The variational form of (2.1) is to find $u\in H_0^1(\Omega)$, $u(x,0)=u_0(x)$ such that

$$(\frac{\partial u}{\partial t},v)+B(u,v)=(f,v) \quad \forall v\in H_0^1(\Omega), \quad t\in[0,T], \tag{2.4}$$

where the bilinear form B is

$$B(u,v)=\int_\Omega\sum_{i,j=1}^{d}a_{ij}\frac{\partial u}{\partial x_j}\frac{\partial v}{\partial x_i}dx \quad \forall u,v\in H^1(\Omega)$$

and

$$(f,v)=\int_\Omega fvdx.$$

We refer the notations of Sobolev space to [5] for details. It is easily seen that the bilinear form $B(u,v)$ is

(1). bounded, i.e. $|B(u,v)|\le C|u|_1|v|_1 \quad \forall u,v\in H_0^1(\Omega)$.

(2). elliptic, i.e. $|B(u,u)|\ge C|u|_1^2 \quad \forall u\in H_0^1(\Omega)$.

We use the backward Euler scheme and Crank-Nicolson scheme for the time discretization [8]. Both schemes are absolutely stable [6]. Let $\triangle t_n$ be the n^{th} time step and M the number of steps, then $\sum_{n=1}^{M}\triangle t_n=T$. We lead to the following problem: for a given function $g_{n-1}\in H^{-1}(\Omega)$, find $w\in H_0^1(\Omega)$ such that

$$A_\tau(w,v)=\tau^{-1}(w,v)+B(w,v)=(g_{n-1},v) \quad \forall v\in H_0^1(\Omega), \tag{2.5}$$

where τ is the time step parameter. For the backward Euler scheme, we have

$$w = u^n - u^{n-1},$$
$$\tau = \triangle t_n,$$
$$(g_{n-1}, v) = (f, v) - B(u^{n-1}, v),$$

and for the Crank-Nicolson scheme, we have

$$w = u^n - u^{n-1},$$
$$\tau = \triangle t_n/2,$$
$$(g_{n-1}, v) = 2((f, v) - B(u^{n-1}, v)).$$

Now we define the τ-norm by

$$\|v\|_\tau^2 = \tau^{-1}(v, v) + B(v, v) \quad \forall v \in H_0^1(\Omega). \tag{2.6}$$

It is known [5] that if Ω is a convex polygon, then for any $g \in L^2(\Omega)$, there exists a solution $u \in H^2(\Omega) \cap H_0^1(\Omega)$ of

$$B(u, v) = (g, v), \quad \forall v \in H_0^1(\Omega)$$

with

$$\|u\|_2 \le C\|g\|_0. \tag{2.7}$$

Here and throughout this paper, c and C (with or without subscript) denote generic positive constants, independent of the time step parameter τ, the mesh parameters L and h_L which will be stated below.

Based on the regularity assumption (2.7), we have

Lemma 2.1. *For any $g \in L^2(\Omega)$, the equation*

$$A_\tau(u, v) = (g, v) \quad \forall v \in H_0^1(\Omega) \tag{2.8}$$

has a solution $u \in H^2(\Omega) \cap H_0^1(\Omega)$ which satisfies

$$\|u\|_2 \le C\|g\|_0. \tag{2.9}$$

Proof. Please refer the proof to [9].

Let Γ_l $(l \ge 0)$ be a quasiuniform triangular partition of Ω with the mesh size $h_l = h_0 2^{-l}$. Γ_l is obtained by linking the midpoints of three edges of triangle on Γ_{l-1}. We assume that $\bar{\Omega} = \cup_{K\in\Gamma_l}\bar{K}$. Let V_l denote the P1 conforming finite element space on Γ_l. Then we obtain the discrete form of (2.8): find $u_l \in V_l$ such that

$$A_\tau(u_l, v_l) = (g, v_l) \quad \forall v_l \in V_l. \tag{2.10}$$

By Lemma 2.1 and the standard finite element estimate, we can easily verify the following [8]

Lemma 2.2. *Assume that u and u_l are the solutions of equations (2.8),(2.10) respectively, then*

$$\|u-u_l\|_\tau \le Ch_l(1+\tau^{-1}h_l^2)^{\frac{1}{2}}\|g\|_0,$$
$$\|u-u_l\|_0 \le Ch_l^2(1+\tau^{-1}h_l^2)\|g\|_0.$$

Define the operator $A_{l,\tau}: V_l \to V_l$ as follows:

$$(A_{l,\tau}u_l, v_l) = A_\tau(u_l, v_l) \quad \forall u_l, v_l \in V_l.$$

Then (2.10) can be expressed by

$$A_{l,\tau}u_l = g_l, \tag{2.11}$$

where $g_l \in V_l$, $(g_l, v) = (g, v) \quad v \in V_l$.

3. Cascadic Multigrid Method

In this section, we will use the cascadic multigrid to solve (2.11) at each time step. Define the cascadic algorithm for (2.11) as follows:

Cascadic Multigrid Method

(1) Set $u_0^0 = u_0^* = u_0$ and let

$$u_l^0 = u_{l-1}^*.$$

(2) For $l = 1, ..., L$

$$u_l^{m_l} = C_{l,\tau}^{m_l} u_l^0.$$

(3) Set $u_l^* = u_l^{m_l}$,

where $C_{l,\tau}$ denote the Richardson iteration procedure, i.e.

$$u_l - C_{l,\tau}^{m_l}u_l^0 = T_{l,\tau}^{m_l}(u_l - u_l^0)$$
$$= (I - R_{l,\tau}A_{l,\tau})(u_l - u_l^0).$$

Here

$$R_{l,\tau} = (\lambda_l + \tau^{-1})^{-1} I,$$

where $\lambda_l = O(h_l^{-2})$.

Following [2], we call a cascadic multigrid method optimal on the level L, if we obtain both the accuracy

$$\|u_L - u_L^*\|_\tau \approx \|u - u_L\|_\tau$$

which means that the iterative error is comparable to the approximation error, and the multigrid complexity

$$amount \;\; of \;\; work = O(n_L),$$

where $n_L = dimV_L$.

Define the operator $B_l : V_l \to V_l$ as follows:

$$(B_l u_l, v_l) = B(u_l, v_l) \quad \forall u_l, v_l \in V_l. \tag{3.1}$$

Let $\{\lambda_i\}_{i=1}^{n_l}$ and $\{\varphi_i\}_{j=1}^{n_l}$ be the eigenvalues and corresponding normalized eigenfunctions of B_l, i.e.

$$B_l \varphi_i = \lambda_i \varphi_i, \quad i = 1, ..., n_l,$$

and

$$(\varphi_i, \varphi_j) = \delta_{ij},$$

where δ_{ij} is Kronecker symbol.

Lemma 3.1. *For the operator $T_{l,\tau}^{m_l}$ and any $v \in V_l$, we have*

$$(1).\|T_{l,\tau}^{m_l} v\|_\tau \le (1 + \tau^{-1}\lambda_l^{-1})^{-m_l} \|v\|_\tau.$$

$$(2).\|T_{l,\tau}^{m_l} v\|_\tau \le C \frac{h_l^{-1}}{m_l^{\frac{1}{2}}} (1 + \tau^{-1}\lambda_l^{-1})^{-\frac{m_l}{2}} \|v\|_0, \quad m_l \ge 2.$$

Proof. We refer the proof of (1) to [1]. We only need to prove (2). It is easy to check that

$$T_{l,\tau}^{m_l} v = \sum_{i=1}^{n_l} C_i \varphi_i (1 - \frac{\lambda_i + \tau^{-1}}{\lambda_l + \tau^{-1}})^{m_l},$$

where $v = \sum_{i=1}^{n_l} C_i \varphi_i$, φ_i is the usual nodal basis on Γ_l. Then

$$\begin{aligned}
\|T_{l,\tau}^{m_l} v\|_\tau^2 &= \sum_{i=1}^{n_l} C_i^2 (\lambda_i + \tau^{-1})(1 - \frac{\lambda_i + \tau^{-1}}{\lambda_l + \tau^{-1}})^{2m_l} \\
&= (\lambda_l + \tau^{-1}) \sum_{i=1}^{n_l} \frac{\lambda_i + \tau^{-1}}{\lambda_l + \tau^{-1}} (1 - \frac{\lambda_i + \tau^{-1}}{\lambda_l + \tau^{-1}})^{m_l - 1} \\
&\qquad \times (1 - \frac{\lambda_i}{\lambda_l})^{m_l + 1} (1 + \tau^{-1}\lambda_l^{-1})^{-m_l - 1} \\
&\le (\lambda_l + \tau^{-1}) \sum_{i=1}^{n_l} (1 + \tau^{-1}\lambda_l^{-1})^{-m_l - 1} \max_{0 \le x \le 1} x(1 - x)^{m_l - 1} \\
&\le C \frac{\lambda_l}{m_l} (1 + \tau^{-1}\lambda_l^{-1})^{-m_l} \|v\|_0^2 \quad m_l \ge 2.
\end{aligned}$$

Now we are able to prove the convergence of the cascadic algorithm for the parabolic problem.

Lemma 3.2. *It holds that*

$$\|u_L - u_L^*\|_\tau \le C \sum_{l=0}^{L} \frac{h_l}{m_l^{\frac{1}{2}}} \prod_{i=0}^{L-l-1} (1 + \tau^{-1}\lambda_{L-i}^{-1})^{-m_{L-i}} (1 + \tau^{-1}\lambda_l^{-1})^{-\frac{m_l}{2} + 1} \|g\|_0, \tag{3.2}$$

where we set $\prod_{i=0}^{-1} = 0$.

Proof. It is easy to check that

$$\begin{aligned}\|u_l-u_l^*\|_\tau &= \|T_{l,\tau}^{m_l}(u_l-u_{l-1}^{m_{l-1}})\|_\tau\\ &\le \|T_{l,\tau}^{m_l}(u_l-u_{l-1})\|_\tau+\|T_{l,\tau}^{m_l}(u_{l-1}-u_{l-1}^{m_{l-1}})\|_\tau\\ &\le C\frac{h_l^{-1}}{m_l^{\frac{1}{2}}}(1+\tau^{-1}\lambda_l^{-1})^{-\frac{m_l}{2}}\|u_l-u_{l-1}\|_0\\ &\quad+(1+\tau^{-1}\lambda_l^{-1})^{-m_l}\|u_{l-1}-u_{l-1}^{m_{l-1}}\|_\tau.\end{aligned}$$

Moreover, by Lemma 2.2 we have

$$\begin{aligned}\|u_l-u_{l-1}\|_0 &\le Ch_l^2(\tau^{-1}h_l^2+1)\|g\|_0\\ &\le Ch_l^2(\tau^{-1}\lambda_l^{-1}+1)\|g\|_0.\end{aligned}$$

Then combining the above two inequalities yields

$$\begin{aligned}\|u_l-u_l^*\|_\tau \le{}& C\frac{h_l}{m_l^{\frac{1}{2}}}(1+\tau^{-1}\lambda_l^{-1})^{-\frac{m_l}{2}+1}\|g\|_0\\ &+(1+\tau^{-1}\lambda_l^{-1})^{-m_l}\|u_{l-1}-u_{l-1}^{m_{l-1}}\|_\tau.\end{aligned}$$

Recurrently, we get

$$\begin{aligned}\|u_L-u_L^*\|_\tau &\le C\frac{h_L}{m_L^{\frac{1}{2}}}(1+\tau^{-1}\lambda_L^{-1})^{-\frac{M_L}{2}+1}\|g\|_0+(1+\tau^{-1}\lambda_L^{-1})^{-M_L}\|u_{L-1}-u_{L-1}^{m_{L-1}}\|_\tau\\ &\le C\frac{h_L}{m_L^{\frac{1}{2}}}(1+\tau^{-1}\lambda_L^{-1})^{-\frac{M_L}{2}+1}\|g\|_0\\ &\qquad+C(1+\tau^{-1}\lambda_L^{-1})^{-M_L}\frac{h_{L-1}}{m_{L-1}^{\frac{1}{2}}}(1+\tau^{-1}\lambda_{L-1}^{-1})^{-\frac{M_{L-1}}{2}+1}\|g\|_0\\ &\qquad+(1+\tau^{-1}\lambda_{L-2}^{-1})^{-m_{L-2}}\|u_{L-2}-u_{L-2}^{m_{L-2}}\|_\tau\\ &\quad\ldots\ldots\\ &\le C\sum_{l=0}^{L}\frac{h_l}{m_l^{\frac{1}{2}}}\prod_{i=0}^{L-l-1}(1+\tau^{-1}\lambda_{L-i}^{-1})^{-m_{L-i}}(1+\tau^{-1}\lambda_l^{-1})^{-\frac{m_l}{2}+1}\|g\|_0.\end{aligned}$$

Note that

$$h_l=h_L2^{L-l}. \tag{3.3}$$

Consider sequences m_1, m_2, ..., m_L of the kind

$$m_l=[\beta^{L-l}m_L] \tag{3.4}$$

for some fixed $\beta>0$, where $[\cdot]$ means the choosing integral function. If τ satisfies

$$\tau\le\lambda_L^{-1},$$

based on the observation in [1], we know that some usual iterative methods, like the Richadson iteration, can already guarantee good convergence. Therefore, we will only consider the case

$$\tau \geq \lambda_L^{-1}.$$

In such case, for any fixed τ, we can find a positive contant $0 < \gamma_0 < 1$ which satisfies

$$\tau \leq \frac{\lambda_L^{-1}}{\gamma_0}, \tag{3.5}$$

where γ_0 is dependent of τ.

Lemma 3.3. *If τ and γ_0 satisfy (3.5), then*

$$\|u_L - u_L^*\|_\tau \leq C\frac{h_L}{m_L^{\frac{1}{2}}}\sum_{l=0}^{L}(\frac{2}{(1+\gamma_0)\beta^{\frac{1}{2}}})^l\|g\|_0, \quad m_L \geq 2.$$

Proof. First, we consider the following term

$$I = \prod_{i=0}^{L-l-1}(1+\tau^{-1}\lambda_{L-i}^{-1})^{-m_{L-i}}(1+\tau^{-1}\lambda_l^{-1})^{-\frac{m_l}{2}+1}. \tag{3.6}$$

It is easy to check that

$$\begin{aligned} I &\leq \prod_{i=0}^{L-l-1}(1+\tau^{-1}\lambda_{L-i}^{-1})^{-m_{L-i}} \qquad m_L \geq 2 \\ &\leq \prod_{i=0}^{L-l-1}\frac{1}{1+\tau^{-1}\lambda_{L-i}^{-1}} \\ &\leq C\prod_{i=0}^{L-l-1}\frac{1}{1+\tau^{-1}\lambda_L^{-1}} \\ &\leq C\frac{1}{(1+\gamma_0)^{L-l}}. \end{aligned} \tag{3.7}$$

Inserting (3.7) to (3.2), we obtain

$$\begin{aligned} \|u_L - u_L^*\|_\tau &\leq C\sum_{l=0}^{L}\frac{h_l}{m_l^{\frac{1}{2}}}\frac{1}{(1+\gamma_0)^{L-l}}\|g\|_0 \\ &\leq C\frac{h_L}{m_L^{\frac{1}{2}}}\sum_{l=0}^{L}(\frac{2}{\beta^{\frac{1}{2}}(1+\gamma_0)})^{L-l}\|g\|_0 \\ &= C\frac{h_L}{m_L^{\frac{1}{2}}}\sum_{l=0}^{L}(\frac{2}{\beta^{\frac{1}{2}}(1+\gamma_0)})^{l}\|g\|_0. \end{aligned}$$

From Lemma 3.3, we obtain the main result of this paper.

Theorem 3.1. *The accuracy of the cascadic multigrid with the Richardson iteration for the parabolic problem is*

$$\|u_L - u_L^*\|_\tau \le C\frac{h_L}{m_L^{\frac{1}{2}}}\frac{1}{1-\frac{2}{\beta^{\frac{1}{2}}(1+\gamma_0)}}\|g\|_0 \quad for \quad \beta > (\frac{2}{1+\gamma_0})^2,$$

where β and m_L are defined in (3.4), and τ is in (3.5).

According to Lemma 1.4 in [2], we have

Lemma 3.4. *The computational cost of the cascadic multigrid is proportional to*

$$\sum_{l=1}^{L} m_l n_l \le C\frac{1}{1-\frac{\beta}{2^d}} m_L n_L \quad for \quad \beta < 2^d. \tag{3.8}$$

Theorem 3.1 indicates that a large β can yield an optimal accuracy. Meanwhile, Lemma 3.4 shows that the optimal complexity of the method can be achieved only for a small β. Therefore, we have

Theorem 3.2. *If β in (3.4) satisfies*

$$(\frac{2}{1+\gamma_0})^2 < \beta < 2^d, \quad d = 1,2,3, \tag{3.9}$$

then both the optimal accuracy and complexity of the cascadic multigrid with the Ricchardson iteration can be obtained.

Remark 3.1. From Theorem 3.2, it is seen that the cascadic multigrid with the Richardson iteration gives the optimal accuracy and complexity for 2D and 3D parabolic problems. But for 1D problem, it requires that the parameter β must be chosen to satisfy

$$(\frac{2}{1+\gamma_0})^2 < \beta < 2,$$

which turns out that the value γ_0 in (3.5) should be greater than $2^{1/2}-1$ that prevents choices of a relatively large time step parameter τ, say of order h in the Crank-Nicolson scheme.

Remark 3.2. Compared with the parabolic case, for 3D elliptic problems, the cascadic multigrid with the Richardson iteration gives the optimal accuracy and complexity. But for the problem in 2D, it gives only nearly optimal complexity. It cannot be used for 1D elliptic problem at all (cf.[2],[7]).

4. Conjugate Gradient Method

Assume that u_l^0 is initial value of CG method on the level l. Let $C_{l,\tau}^{m_l}u_l^0$ be the m_l steps of CG iteration. Then the error of CG method can be expressed by

$$\|u_l - C_{l,\tau}^{m_l}u_l^0\|_\tau = \min_{p\in P_{m_l}, p(0)=1} \|p(A_{l,\tau})(u_l - u_l^0)\|_\tau, \tag{4.1}$$

where P_{m_l} denotes the set of polynomials p with degree $p \le m_l$ (cf. [3]).

Using a same argument of Theorem 2.2 in [2], we have

Lemma 4.1. *There exists a linear operator* $T_{l,\tau} = \phi_{\lambda_l,m_l}(A_{l,\tau})$, *where* $\phi_{\lambda,m} \in P_m$, $\phi_{\lambda,m}(0) = 1$ *such that*

$$(1).\|T_{l,\tau}^{m_l} v_l\|_\tau \le \frac{(\lambda_l + \tau^{-1})^{\frac{1}{2}}}{2m_l + 1}\|v_l\|_0 \quad \forall v_l \in V_l, \tag{4.2}$$

$$(2).\|T_l^{m_l} v_l\|_\tau \le \|v_l\|_\tau \quad \forall v_l \in V_l. \tag{4.3}$$

Using Lemma 4.1 and following the same line of Lemma 1.3 in [2], we have

Lemma 4.2. *Assume that the time step parameter* $\tau \ge O(h_L^2)$, *then the accuracy of the cascadic multigrid with CG method as smoother is*

$$\|u_L - u_L^*\|_\tau \le \begin{cases} C\dfrac{1}{1-(\frac{2}{\beta})}\dfrac{h_L}{m_L}\|g\|_0 & for \quad \beta > 2, \\ CL\dfrac{h_L}{m_L}\|g\|_0 & for \quad \beta = 2. \end{cases} \tag{4.4}$$

Remark 4.1. *It should be noticed that the assumption on the time step parameter* $\tau \ge O(h_L^2)$ *in Lemma 4.2 is not a real restriction since we can always assume* $\tau = O(h_L^2)$ *for the backward Euler scheme and* $\tau = O(h_L)$ *for the Crank-Nicolson Scheme. Moreover, as pointed out in [1], for a small time step parameter* $\tau \le O(h_L^2)$, *some standard iterative methods alone are efficient enough to guarantee a good convergence.*

Combining Lemma 4.2 with Lemma 1.4 in [2], we get

Theorem 4.1. *(1). For 2D, 3D parabolic problems, the optimal accuracy and complexity can be obtained for the cascadic multigrid with CG iteration.*

(2). For 1D parabolic problems, if we choose $\beta = 2$ *and the number of iterations on the level L is*

$$m_L = [m_* L],$$

then the error of the cascadic multigrid method is

$$\|u_L - u_L^*\|_\tau \le C\frac{h_L}{m_*}\|g\|_0$$

and the complexity of computation is

$$\sum_{l=1}^{L} m_l n_l \le c m_* n_L (1 + log n_L)^2.$$

Remark 4.2. Besides the P_1 conforming finite element, the above results are also valid for other conforming or nonconforming finite elements of the second order problem (cf. [7]).

Remark 4.3. In practical computation, the right hand term g_l in (2.11) is related to the cascadic multigrid solution of the previous time step. According to [6], the backward

Euler and Crank-Nicolson schemes are absolutely stable, so the small perturbation of the right hand term in (2.11) still assure the efficiency of our algorithm.

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