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# A NOTE ON THE FINITE ELEMENT METHOD FOR THE REISSNER-MINDLIN PLATE*1)

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**Abstract**

In this short note we use the idea of R.Duran [5] and introduce a new low-order triangular element by replacing the nonconforming linear element for the rotation in [2] with the conforming linear element.The optimal order error estimates are obtained uniformly in the plate thickness.

## 1. Introduction

We consider the finite element method for the Reissner-Mindlin plate model in the primitive variables. The plate model describes the displacement of a plate with small to moderate thickness subject to a transverse load.

It is well known that the standard finite element method fails to give good approximations when the plate thickness is too small, owing to a locking phenomenon. In 1986, Brezzi and Fortin[1] introduced an equivalent of the Reissner-Mindlin plate model by using a Helmhotz decomposition of shear strain. Recently, Arnold and Falk[2] developed the idea of [1] and proposed a low-order triangular element in the primitive variables by introducing a discrete version of the Helmholtz decomposition. They analyzed and proved uniform optimal-order convergence in the plate thickness. To remove the internal degrees of freedom in the Arnold-Falk element [2], R.Duran[3] introduced a modification of the element in [2] and proved the uniform optimal order convergence with respect to thickness.

In 1992, Cheng [6] proposed a simple finite element method for the Reissner-Mindlin plate, by using conforming linear element both the transverse displacement and rotation by a new discrete version of the Helmholtz decomposition. It was proved that the method converges with optimal order uniformly with respect to thickness.

In this short note we use the idea of the mixed finite element method proposed by R.Duran et al [5] and introduce a new low-order triangular element, in which the

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nonconforming linear element are replaced by the conforming linear element the Arnold-Falk element [2]. Our analysis provides a direct proof of the convergence without using the discrete Helmhotz decomposition. To eliminate the bubble function of displacement at element level, we obtain the Cheng element [6].

## 2. The Reissner-Mindlin plate model and notation

We use standard notation for the Sobolev spaces $H^k(\Omega)$ and $H_0^k(\Omega)$ with the norm

$$\|u\|_k^2 = \sum_{|\alpha|\le k} \|D^\alpha u\|_{L^2(\Omega)}^2$$

for both scalar and vector functions. Boldface type is used to denote vector spaces.

Let $\Omega \times (-\frac{t}{2}, \frac{t}{2})$ be the region occupied by the undeformed plate, where $\Omega \subset R^2$ is a simply connected polygon and $0 < t < 1$ is the plate thickness. For the sake of simplicity, we consider the following variational Reissner-Mindlin plate.

**Problem RM.** Find $(\omega, \vec{\beta}) \in H_0^1(\Omega) \times \boldsymbol{H}_0^1(\Omega)$ such that

$$a(\vec{\beta}, \vec{\eta}) + t^{-2}(\nabla w - \vec{\beta}, \nabla v - \vec{\eta}) = (g, v), \quad \forall (v, \vec{\eta}) \in H_0^1(\Omega) \times \boldsymbol{H}_0^1(\Omega). \tag{2.1}$$

Here $(\cdot,\cdot)$ denotes the scalar product in either $L^2(\Omega)$ or $\boldsymbol{L}^2(\Omega)$, and the bilinear form $a(\cdot,\cdot)$ is defined as in [1-6]. It is known that $a(\cdot,\cdot)$ defines an inner product in $\boldsymbol{H}_0^1(\Omega)$ equivalent to the usual one.

Let

$$\vec{\nu} = t^{-2}(\nabla w - \vec{\beta}) \tag{2.2}$$

be the shear strain. We use $C$ to denote a constant independent of $h$ and $t$ but not necessarily the same at each occurrence.

## 3. The finite element method and error analysis

Let $\Omega$ be a convex polygon and $\mathcal{T}_h$ be a regular triangulation of $\Omega$[7], where as usual $h$ stands for the mesh size. Denoting $P_k(T)$ as the set of functions on $T$ which are polynomials of degree no greater than $k$, we define the following finite element spaces:

$$W_h = \{w \in H_0^1(\Omega) : w|_T \in P_1(T), \forall T \in \mathcal{T}_h\}, \tag{3.1}$$

$$\overline{\Gamma}_h = \{\vec{\nu} \in \boldsymbol{L}^2(\Omega) : \vec{\nu}|_T \in \boldsymbol{P}_0(T), \forall T \in \mathcal{T}_h\}, \tag{3.2}$$

$$\boldsymbol{H}_h = \{\vec{\eta} \in \boldsymbol{H}_0^1(\Omega) : \vec{\eta}|_T \in \boldsymbol{P}_1(T) \oplus \boldsymbol{P}_0(T) b_T, \forall T \in \mathcal{T}_h\}, \tag{3.3}$$

where $b_T$ is a bubble function of degree 3, namely, $b_T \in P_3(T)$ and $b_T = 0$ on $\partial T$.

Define the orthogonal projection $\overline{\Pi}$ :

$$\boldsymbol{L}^2(\Omega) \to \overline{\Gamma}_h \quad \forall \vec{\nu} \in \boldsymbol{L}^2(\Omega), \quad \overline{\Pi}\vec{\nu} \in \overline{\Gamma}_h$$

and

$$\Pi \vec{\nu}|_T = \frac{1}{\text{meas}(T)} \int_T \vec{\nu}\, dx. \tag{3.4}$$

Obviously,

$$\|\vec{\nu} - \Pi \vec{\nu}\|_0 \le Ch\|\vec{\nu}\|_1, \text{ for } \vec{\nu} \in \boldsymbol{H}^1(\Omega), \tag{3.5}$$

$$\|\Pi \vec{\nu}\|_0 \le \|\vec{\nu}\|_0, \text{ for } \vec{\nu} \in \boldsymbol{L}^2(\Omega). \tag{3.6}$$

Then our approximation scheme is given in the following problem.

**Problem CW.** Find $(\omega_h, \vec{\beta}_h) \in W_h \times \boldsymbol{H}_h$ such that

$$a(\vec{\beta}_h, \vec{\eta}) + t^{-2}(\nabla w_h - \Pi \vec{\beta}_h, \nabla v - \Pi \vec{\eta}) = (g, v), \quad \forall (v, \vec{\eta}) \in W_h \times \boldsymbol{H}_h. \tag{3.7}$$

The existence and uniqueness of the solution follow easily from the coerciveness of $a(\cdot,\cdot)$. Denote

$$\vec{\nu}_h = t^{-2}(\nabla w_h - \Pi \vec{\beta}_h). \tag{3.8}$$

From Duran et al [5], we have

**Lemma 3.1**[5]. *For any* $\vec{\hat{\beta}} \in \boldsymbol{H}_h, \hat{w} \in W_h$ *and* $\vec{\hat{\nu}} = t^{-2}(\nabla \hat{w} - \Pi \vec{\hat{\beta}})$,

$$\|\vec{\hat{\beta}} - \vec{\beta}_h\|_1 + t\|\vec{\hat{\nu}} - \vec{\nu}_h\|_0 \le C\left\{\|\vec{\hat{\beta}} - \vec{\beta}\|_1 + t\|\vec{\hat{\nu}} - \vec{\nu}\|_0 + h\|\vec{\nu}\|_0\right\}. \tag{3.9}$$

From Lemma 3.1 we see that, if there exist $\vec{\hat{\beta}} \in \boldsymbol{H}_h$ and $\hat{w} \in W_h$ such that $\vec{\hat{\beta}}$ and $\vec{\hat{\nu}}$ are good approximations of $\vec{\beta}$ and $\vec{\nu}$, respectively, we get an error estimate. The key is to choose $\vec{\hat{\beta}}$ and $\hat{w}$ properly. Duran et al[5] first chose $\vec{\hat{\beta}}$ and then found $\hat{w}$ such that $\vec{\hat{\nu}}$ is a good approximation of $\vec{\nu}$. Although they introduced a low-order triangular element by using the quadratic element for displacement, it was not optimal for displacement and was not economical. Here we first choose $\hat{w}$ and then find $\vec{\hat{\beta}}$ such that $\vec{\hat{\nu}}$ is a good approximation of $\vec{\nu}$. We obtain the optimal-order error estimate for our method (3.7).

Let $\Pi_1 : H_0^1(\Omega) \to W_h$ be the orthogonal projection and $\overline{\Pi}_1 = \Pi_1 \times \Pi_1$. Then, from [7],

$$\|w - \Pi_1 w\|_0 + h\|w - \Pi_1 w\|_1 \le Ch^2\|w\|_2, \tag{3.10}$$

$$\|\vec{\eta} - \overline{\Pi}_1 \vec{\eta}\|_0 + h\|\vec{\eta} - \overline{\Pi}_1 \vec{\eta}\|_1 \le Ch^2\|\vec{\eta}\|_2 \tag{3.11}$$

for any $w \in H^2(\Omega) \cap H_0^1(\Omega), \vec{\eta} \in \boldsymbol{H}^2(\Omega) \cap \boldsymbol{H}_0^1(\Omega)$.

Let $(w, \vec{\beta})$ be the solution of Problem RM. We first choose

$$\hat{w} = \Pi_1 w \in W_h. \tag{3.12}$$

Then we define $\vec{\hat{\beta}} \in \boldsymbol{H}_h$ such that

$$\text{(i)} \quad \vec{\hat{\beta}} = \overline{\Pi}_1 \vec{\beta} + R\vec{\beta}, \qquad \text{(ii)} \ \Pi R|_T = \vec{\alpha}_T b_t, \forall T \in \mathcal{T}_h, \tag{3.13}$$

$$\text{(iii)} \quad \vec{\alpha}_T = 60\Pi\{\nabla(w - \Pi_1 w) - (\vec{\beta} - \overline{\Pi}_1 \vec{\beta})\}|_T.$$

Obviously, from (3.10), (3.11) and (3.6),

$$|\vec{\alpha}_T| \leq C\{\|\nabla(w - \Pi_1 w)\|_{0,T} + \|\vec{\beta} - \Pi_1\vec{\beta}\|_{0,T}\},$$

$$\|\vec{\beta} - \hat{\vec{\beta}}\|_1 \leq \|\vec{\beta} - \Pi_1\vec{\beta}\|_1 + \|R\vec{\beta}\|_1 \leq C\Big\{h\|\vec{\beta}\|_2 + \Big(\sum_{T\in\mathcal{T}_h} |\vec{\alpha}|^2\Big)^{\frac{1}{2}}\Big\}$$

$$\leq Ch(\|\vec{\beta}\|_2 + \|w\|_2). \tag{3.14}$$

From Lemma 3.1, $\hat{\vec{\nu}} = t^{-2}(\nabla\hat{w} - \overline{\Pi}\hat{\vec{\beta}}) \in \overline{\Gamma}_h$. For any $T \in \mathcal{T}_h$,

$$(\overline{\Pi}\vec{\nu} - \hat{\vec{\nu}})|_T = \{t^{-2}(\overline{\Pi}\nabla w - \overline{\Pi}\vec{\beta}) - t^{-2}(\nabla\Pi_1 w - \overline{\Pi}(\Pi_1\vec{\beta} + R\vec{\beta}))\}|_T$$
$$= t^{-2}\{\overline{\Pi}(\nabla w - \nabla\Pi_1 w)|_T - \overline{\Pi}(\vec{\beta} - \Pi_1\vec{\beta})|_T - \frac{1}{60}\vec{\alpha}_T\} = 0.$$

Then, $\hat{\vec{\nu}} = \overline{\Pi}\vec{\nu}$ and by (3.5),

$$\|\vec{\nu} - \hat{\vec{\nu}}\|_0 \leq Ch\|\vec{\nu}\|_1. \tag{3.15}$$

Using the error estimates (3.14),(3.15) and Lemma 3.1, we get

$$\|\vec{\beta} - \vec{\beta}_h\|_1 \leq \|\vec{\beta} - \hat{\vec{\beta}}\|_1 + \|\hat{\vec{\beta}} - \vec{\beta}_h\|_1 \leq Ch(\|\vec{\beta}\|_2 + t\|\vec{\nu}\|_1 + \|\vec{\nu}\|_0).$$

Applying the regularity result[1,2], we have

**Theorem 3.1.** *Let $(w, \vec{\beta})$ and $(w_h, \vec{\beta}_h)$ be the solutions of Problem RM and Problem CW, respectively. If $g \in L^2(\Omega)$ , then we have*

$$\|\vec{\beta} - \vec{\beta}_h\|_1 \leq Ch\|g\|_0. \tag{3.16}$$

**Remark 1.** The error estimate $\|w - w_h\|_1$ can be derived easily by (3.15) and Corollary 3.1 of [5].

**Remark 2.** If we eliminate the bubble function $b_t$ at element level [3,4], then we get the Cheng element [6] directly without using the discrete version of Helmholtz decomposition.

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