Journal of Computational Mathematics, Vol.11, No.2, 1993, 178-187.

THE MULTIGRID METHOD FOR TRUNC PLATE ELEMENT*

Wang Ming
(*Department of Mathematics, Peking University, Beijing, China*)

Abstract

This paper develops an optimal-order multigrid method for the TRUNC plate element.

This paper will consider the multigrid method for the TRUNC element proposed by Bergan et al. and developed further by Argyris et al. The numerical experiences show that the element has very good convergence(cf. [1,2]). The mathematical proof of convergence of the TRUNC element is also given by Shi Zong-ci in [7]. An optimal multigrid method for the element is given in this paper and the method consists of presmoothing and correction on coarser grids.

§1. The TRUNC Element

Given a triangle K with vertices $a_i = (x_i, y_i), i = 1,2,3$, we denote by λ_i the area coordinates for the triangle and put

$$\xi_1 = x_2 - x_3, \quad \xi_2 = x_3 - x_1, \quad \xi_3 = x_1 - x_2,$$
$$\eta_1 = y_2 - y_3, \quad \eta_2 = y_3 - y_1, \quad \eta_3 = y_1 - y_2.$$

The nodal parameters of the element are the function values and the values of the two first derivatives at the vertices of the triangle K. According to [7], on the triangle K the shape function is an incomplete cubic polynomial,

$$\begin{aligned} w =& b_1\lambda_1 + b_2\lambda_2 + b_3\lambda_3 + b_4\lambda_1\lambda_2 + b_5\lambda_2\lambda_3 + b_6\lambda_3\lambda_1 + b_7(\lambda_1^2\lambda_2 - \lambda_1\lambda_2^2) \\ &+ b_8(\lambda_2^2\lambda_3 - \lambda_2\lambda_3^2) + b_9(\lambda_3^2\lambda_1 - \lambda_3\lambda_1^2), \end{aligned} \tag{1.1}$$

which is uniquely determined by the nine nodal parameters $w_i, w_x(i), w_y(i), i = 1,2,3$.

* Received June 15, 1991.

The coefficients b_i are determined as follows:

$$\begin{cases} b_i = w_i, \quad i = 1,2,3, \\ b_4 = -\dfrac{1}{2}\{(w_x(1) - w_x(2))\xi_3 + (w_y(1) - w_y(2))\eta_3\}, \\ b_5 = -\dfrac{1}{2}\{(w_x(2) - w_x(3))\xi_1 + (w_y(2) - w_y(3))\eta_1\}, \\ b_6 = -\dfrac{1}{2}\{(w_x(3) - w_x(1))\xi_2 + (w_y(3) - w_y(1))\eta_2\}, \\ b_7 = w_1 - w_2 - \dfrac{1}{2}(w_x(1) + w_x(2))\xi_3 - \dfrac{1}{2}(w_y(1) + w_y(2))\eta_3, \\ b_8 = w_2 - w_3 - \dfrac{1}{2}(w_x(2) + w_x(3))\xi_1 - \dfrac{1}{2}(w_y(2) + w_y(3))\eta_1, \\ b_9 = w_3 - w_1 - \dfrac{1}{2}(w_x(3) + w_x(1))\xi_2 - \dfrac{1}{2}(w_y(3) + w_y(1))\eta_2. \end{cases} \tag{1.2}$$

The shape form (1.1) with (1.2) is another Zienkiewicz's element. This element is a C^0 element,, nonconforming for plate bending problems, which converges to the true solution only for very special meshes. The TRUNC element is obtained by modifying the variational formulation.

Let Ω be a convex polygon in $R^2, f \in L^2(\Omega)$. Consider the plate bending problem with the clamped boundary conditions,

$$\begin{cases} \triangle^2 u = f, \quad \text{in} \quad \Omega, \\ u|_{\partial\Omega} = \dfrac{\partial u}{\partial N}\Big|_{\partial\Omega} = 0. \end{cases} \tag{1.3}$$

The weak form of the problem (1.3) is to find $u \in H_0^2(\Omega)$ such that

$$a(u,v) = (f,v), \quad \forall v \in H_0^2(\Omega), \tag{1.4}$$

where

$$\begin{aligned} a(u,v) &= \int_\Omega (\triangle u \triangle v + (1-\sigma)(2u_{xy}v_{xy} - u_{xx}v_{yy} - u_{yy}v_{xx}))\, dxdy, \\ (f,v) &= \int_\Omega fv\, dxdy, \end{aligned} \tag{1.5}$$

and $0 < \sigma < \frac{1}{2}$ is the Poisson ratio.

Let $\{\mathcal{K}^k\}_{k=1}^\infty$ be a family of subdivisions of Ω by triangles, where $\mathcal{K}^{k+1}$ is obtained by connecting midpoints of the edges of the triangles in $\mathcal{K}^k$. Let $h_k = \max_{K \in \mathcal{K}^k} \operatorname{diam} K$. Then $h_{k-1} = 2h_k$ and there exist positive constants C_1, C_2, independent of k, such that

$$C_2 h_k^2 \le |K| \le C_1 h_k^2, \quad \forall K \in \mathcal{K}^k, \tag{1.6}$$

where $|K|$ is the area of the triangle K. Throughout the paper, C with or without subscript denotes generic positive constants independent of k.

For $k = 1, 2, \cdots$, defining on each triangle $K \in \mathcal{K}^k$ the shape function in the form of

(1.1) and (1.2), we obtain the finite element space V_k. Define, for $w, v \in H_0^2(\Omega) + V_k$,

$$a_k(w,v) = \sum_{K\in\mathcal{K}^k} \int_K [\triangle w \triangle v + (1-\sigma)(2w_{xy}v_{xy} - w_{xx}v_{yy} - w_{yy}v_{xx})]\, dxdy. \tag{1.7}$$

For $v_k \in V_K$, it can be written in two parts,

$$v_k = \bar{v}_k + v_k' \tag{1.8}$$

with

$$\begin{cases} \bar{v}_k|_K = b_1\lambda_1 + b_2\lambda_2 + b_3\lambda_3 + b_4\lambda_1\lambda_2 + b_5\lambda_2\lambda_3 + b_6\lambda_3\lambda_2, \\ v_k'|_K = b_7(\lambda_1^2\lambda_2 - \lambda_1\lambda_2^2) + b_8(\lambda_2^2\lambda_3 - \lambda_2\lambda_3^2) + b_9(\lambda_3^2\lambda_1 - \lambda_3\lambda_1^2). \end{cases} \tag{1.9}$$

Now we define another bilinear form on V_k,

$$b_k(w,v) = a_k(\bar{w},\bar{v}) + a_k(w',v'), \quad \forall w, v \in V_k. \tag{1.10}$$

The mathematical description of the TRUNC element is to find $u_k \in V_k$ such that

$$b_k(u_k, v) = (f, v), \quad \forall v \in V_k. \tag{1.11}$$

For $0 \le m \le 3$, define semi-norm $|\cdot|_{m,k}$ as follows:

$$|\cdot|_{m,k}^2 = \sum_{K\in\mathcal{K}^k} |\cdot|_{m,K}^2, \tag{1.12}$$

where $|\cdot|_{m,K}$ is the Sobolev semi-norm of $H^m(K)$. From Lemma 3.3 in [7], we have

$$C|v|_{2,k}^2 \le b_k(v,v), \quad \forall v \in V_k, \tag{1.13}$$

$$|\bar{v}|_{2,k} + |v'|_{2,k} \le C|v|_{2,k}, \quad \forall v \in V_k. \tag{1.14}$$

Let $w \in H^3(\Omega)$ and $\Pi_k w$ be the interpolation of w. Then from Lemma 3.2 in [7], the following estimates are true for $m = 0, 1, 2, 3$:

$$\begin{cases} |w - \Pi_k w|_{m,k} \le C h_k^{3-m} |w|_3, \\ |w - \overline{\Pi_k w}|_{m,k} \le C h_k^{3-m} |w|_3, \\ |(\Pi_k w)'|_{m,k} \le C h_k^{3-m} |w|_3, \end{cases} \tag{1.15}$$

where $|w|_3$ is the Sobolev semi-norm of w.

From Theorems 3.8 and 3.12 in [7], we have

$$\|u - u_k\|_0 + h_k|u - u_k|_{2,k} \le h_k^2 |u|_3, \tag{1.16}$$

where u is the solution of (1.4) and u_k is that of (1.11).

Define $D = (\partial_x, \partial_y)^\top$. For $v_k, w_k \in V_k$, define

$$(v_k, w_k)_{0,k} = \sum_{K\in\mathcal{K}^k} \sum_{i=1}^{3} [v_k(a_i)w_k(a_i) + h_k^2 Dv_k(a_i)Dw_k(a_i)]h_k^2, \tag{1.17}$$

$$|||v_k|||_{0,k} = (v_k, v_k)_{0,k}^{\frac{1}{2}}, \tag{1.18}$$

$$|[v_k]|_k^2 = \sum_{K\in\mathcal{K}^k} \sum_{\substack{1\le i,j\le 3\\ i\ne j}} (|v_k(a_i) - v_k(a_j) - Dv_k(a_i)(a_i - a_j)|^2$$
$$+ (Dv_k(a_i) - Dv_k(a_j))(Dv_k(a_i) - Dv_k(a_j))h_k^2)h_k^{-2}. \tag{1.19}$$

It is easy to show that the following inequalities are true for $\forall v_k \in V_k$:

$$C_4|||v_k|||_{0,k} \le \|v_k\|_0 \le C_3|||v_k|||_{0,k}, \tag{1.20}$$

$$C_6|[v_k]|_k \le |v_k|_{2,k} \le C_5|[v_k]|_k. \tag{1.21}$$

§2. The Multigrid Method

Since $(\cdot,\cdot)_{0,k}$ is an inner product on V_k, there exist, from the spectral theory, eigenvalues $0 < \mu_1 \le \mu_2 \le \cdots \le \mu_{n_k}$ and eigenfunctions $\psi_1, \psi_2, \cdots, \psi_{n_k} \in V_k, (\psi_i, \psi_j)_{0,k} = \delta_{ij}$ (=the Kronecker delta), such that $b_k(\psi_i, v) = \mu_i(\psi_i, v)_{0,k}$ for all $v \in V_k$. From the inverse estimates, there exists a constant C_0 such that

$$\mu_{n_k} \le C_0 h_k^{-4}. \tag{2.1}$$

For $v_k \in V_k$, we can write $v_k = \sum_{i=1}^{n_k} \nu_i\psi_i$. The norm $|||v_k|||_{s,k}$ is defined as follows:

$$|||v_k|||_{s,k} = \left(\sum_{i=1}^{n_k} \nu_i^2\mu_i^{s/2}\right)^{\frac{1}{2}}. \tag{2.2}$$

Obviously, $|||v_k|||_{2,k}^2 = b_k(v_k, v_k) \le C|v_k|_{2,k}^2$. From the Schwarz inequality, we have

$$b_k(v, w) \le |||v|||_{2+t,k}|||w|||_{2-t,k} \tag{2.3}$$

for all $t \in (-\infty, \infty)$ and $v, w \in V_k$.

2.1. The intergrid transfer operator. To develop a multigrid method we have to choose an intergrid transfer operator $I_{k-1}^k : V_{k-1} \to V_k$ because $V_{k-1} \not\subseteq V_k$.

For $v \in V_{k-1}$, $I_{k-1}^k v$ is defined as follows. For $K \in \mathcal{K}^{k-1}$, denote the four triangles obtained by connecting the midpoints of the edges of K by $K_i, 1 \le i \le 4$. Let b_1, b_2, b_3 be the vertices of K_i. Then $I_{k-1}^k v(b_j) = v(b_j)$ for $j = 1, 2, 3$. If b_j is also a vertex of K, then $DI_{k-1}^k v(b_j) = Dv(b_j)$. If b_j is the midpoint of the edge $\overline{a_l a_k}$ of K, then $DI_{k-1}^k v(b_j) = \frac{1}{2}(Dv(a_k) + Dv(a_l))$.

2.2. The kth-level iteration. The kth-level iteration with initial guess z_0 yields $MG(k, z_0, G)$ as an approximate solution to the following problem:

$$\text{(P)} \quad \text{Find } z \in V_k, \text{ such that } \quad b_k(z, v) = G(v),\ \forall v \in V_k, \text{ where } G \in V_k'. \tag{2.4}$$

For $k = 1$, $MG(1, z_0, G)$ is the solution obtained from a direct method. For $k > 1$, there are two steps.

Pre-smoothing Step. Let $z_i \in V_k$ $(1 \le i \le m)$ be defined recursively by the equations

$$(z_i - z_{i-1}, v)_{0,k} = \frac{1}{\Lambda_k}(G(v) - b_k(z_{i-1}, v)), \quad \forall v \in V_k. \tag{2.5}$$

Here $\Lambda_k = C_0 h_k^{-4}$ and m is a nonnegative integer to be determined later.

Correction Step. Let $\bar{G} \in V'_{k-1}$ be defined by

$$\bar{G} \equiv G(I_{k-1}^k v) - b_k(z_m, I_{k-1}^k v) = b_k(z - z_m, I_{k-1}^k v), \quad \forall v \in V_{k-1}.$$

Let $q_i \in V_{k-1} (0 \le i \le t, t = 2$ or $3)$ be defined recursively by

$$q_0 = 0, \quad q_i = MG(k-1, q_{i-1}, \bar{G}).$$

Let $z_{m+1} = z_m + I_{k-1}^k q_t$. Then $MG(k, z_0, G)$ is defined to be z_{m+1}.

2.3. The full multigrid algorithm. For $k = 1$, the approximate solution $\tilde{u}_1$ of (1.11) is obtained by a direct method. For $k = j$ with $j \ge 2$, the approximate solutions $\tilde{u}_j$ are obtained recursively by

$$u_0^j = I_{j-1}^j \tilde{u}_{j-1},$$
$$u_l^j = MG(j, u_{l-1}^j, F), \quad 1 \le l \le r, \quad F(v) = \iint_\Omega f v^0 dx dy,$$
$$\tilde{u}_j = u_r^j.$$

where r is a positive integer to be determined.

§3. Convergence Analysis of the Multigrid Method

To analyze the multigrid algorithm, we first give some estimates about the intergrid transfer operator I_{k-1}^k.

Lemma 1. *There exists a constant C independent of k, such that*

$$\|I_{k-1}^k v\|_0 \le C \|v\|_0, \tag{3.1}$$

$$C^{-1} |v|_{2,k-1} \le C |I_{k-1}^k v|_{2,k} \le |v|_{2,k-1}, \tag{3.2}$$

$$\sum_{i=0}^{2} h_k^i |v - I_{k-1}^k v|_{i,k} \le C h_k^3 |v|_{3,k-1} \tag{3.3}$$

for $\forall v \in V_{k-1}$ and $k \ge 2$.

Proof. From the definition of I_{k-1}^k and inequalities (1.17) to (1.21), we can get (3.1) and (3.2) easily. Let $v \in V_{k-1}$ and K be a triangle in $\mathcal{K}^{k-1}$. If $v|_K \in P_2(K)$, then $v|_K = I_{k-1}^k v|_K$. Noticing that $I_{k-1}^k v|_K$ is uniquely determined by $v|_K$, we get inequality (3.3) from the interpolation theory (cf. [5]).

Lemma 2. *If $w \in H^3(\Omega) \cap H_0^2(\Omega)$, then for $k \ge 2$,*

$$\sum_{i=0}^{2} h_k^i |w - I_{k-1}^k \Pi_{k-1} w|_{i,k} \le C h_k^3 |w|_3. \tag{3.4}$$

Proof. Let $0 \le i \le 2$. Then from (3.3) and (1.15) we have

$$|w - I_{k-1,K_i}^k \Pi_K w|_{i,k} \le |w - \Pi_{k-1} w|_{i,k-1} + |\Pi_{k-1} w - I_{k-1}^k \Pi_{k-1} w|_{i,k}$$
$$\le C(h_{k-1}^{3-i} |w|_3 + h_k^{3-i} |\Pi_{k-1} w|_{3,k-1}).$$

From the fact that $h_{k-1} = 2h_k$ and $|\Pi_{k-1}w|_{3,k-1} \le C|w|_3$, inequality (3.4) follows.

Lemma 3. *For any $v \in V_k$, we have*

$$|||v|||_{1,k} \le C|v|_1. \tag{3.5}$$

Proof. Let $P : L^2(\Omega) \to V_k$ be the $L^2(\Omega)$ orthogonal projection operator, i.e., for $\forall w \in L^2(\Omega)$,

$$(Pw, v) = (w, v), \forall v \in V_k.$$

Then we have

$$|||Pw|||_{0,k} \le C\|Pw\|_0 \le C\|w\|_0, \quad \forall w \in L^2(\Omega),$$

$$\|w - Pw\|_0 \le C\|w\|_0, \quad \forall w \in L^2(\Omega).$$

From (1.15), we get

$$\|w - Pw\|_0 \le Ch_k^3\|w\|_3, \quad \forall w \in H_0^3(\Omega).$$

Then from the interpolation theory of the Hilbert spaces in [6], we have

$$\|w - Pw\|_0 \le Ch_k^2\|w\|_2, \quad \forall w \in H_0^2(\Omega).$$

For each K in $\mathcal{K}^k$, define $P_K : L^2(K) \to P_5(K)$ as the $L^2(K)$ orthogonal projection operator. Then

$$\sum_{i=0}^{2} h_k^i|v - P_K v|_{i,K} \le h_k^2\|v\|_{2,K}, \quad \forall v \in H^2(K).$$

From the above two inequalities and the inverse estimates, we have, for $w \in H_0^2(\Omega)$,

$$\begin{aligned}
|||Pw|||_{2,k}^2 &\le C|Pw|_{2,k}^2 \le C\sum_{K\in\mathcal{K}^k}(|Pw - P_K w|_{2,K}^2 + |P_K w|_{2,K}^2)\\
&\le C\sum_{K\in\mathcal{K}^k}(h_k^{-4}|Pw - P_K w|_{0,K}^2 + |w|_{2,K}^2)\\
&\le C\Big\{\sum_{K\in\mathcal{K}^k} h_k^{-4}(|w - P_K w|_{0,K}^2 + |w - Pw|_{0,K}^2) + |w|_2^2\Big\}\\
&\le C\Big\{h_k^{-4}(|w - Pw|_0^2 + \sum_{K\in\mathcal{K}^k}|w - P_K w|_{0,K}^2) + |w|_2^2\Big\}\\
&\le C\Big\{\sum_{K\in\mathcal{K}^k}|w|_{2,K}^2 + |w|_2^2\Big\} = C|w|_2^2.
\end{aligned}$$

From the interpolation theory of Hilbert spaces (cf.[6]), we have

$$|||Pw|||_{1,k} \le C|w|_1, \quad \forall w \in H_0^1(\Omega).$$

Inequality (3.5) follows from the fact that for $\forall v \in V_k$, $v \in H_0^1(\Omega)$ and $Pv = v$.

Now we turn to the convergence analysis of the kth-level iteration. Let z be the exact solution of (P). Let $e_i = z - z_i, 0 \le i \le m+1$. Then $e_{m+1} = z - z_{m+1} = e_m - I_{k-1}^k q_t$.

Let $q \in V_{k-1}$ satisfy

$$b_{k-1}(q, v) = \bar{G}(v) \equiv b_k(e_m, I_{k-1}^k v), \quad \forall v \in V_{k-1}. \tag{3.6}$$

Define $\tilde{z}_{m+1} = z_m + I_{k-1}^k q$ and $\tilde{e}_{m+1} = z - \tilde{z}_{m+1}$.

Lemma 4.

$$|e_m|_{2,k} \le C|e_0|_{2,k}, \tag{3.7}$$

$$|||e_m|||_{3,k} \le Ch_k^{-1}(1+m)^{-1/4}|e_0|_{2,k}. \tag{3.8}$$

Proof. Inequality (3.7) follows from the fact that in each relaxation step $|||\cdot|||_{s,k}$ is not increased. The proof of inequality (3.8) is similar to the one in [3].

Lemma 5.

$$|\tilde{e}_{m+1}|_{2,k} \le Ch_k|||e_m|||_{3,k}. \tag{3.9}$$

Proof. Let $\tilde{q} \in V_{k-1}$ satisfy

$$b_k(I_{k-1}^k\tilde{q}, I_{k-1}^k v) = b_k(e_m, I_{k-1}^k v), \quad \forall v \in V_{k-1}. \tag{3.10}$$

From inequality (3.2), we see that $\tilde{q}$ exists uniquely. From (3.5) and (3.10) we have

$$\begin{aligned} b_k(e_m - I_{k-1}^k\tilde{q}, e_m - I_{k-1}^k\tilde{q}) &= b_k(e_m, e_m - I_{k-1}^k\tilde{q}) \\ &\le |||e_m|||_{3,k}|||e_m - I_{k-1}^k\tilde{q}|||_{1,k} \le C|||e_m|||_{3,k}|e_m - I_{k-1}^k\tilde{q}|_1. \end{aligned} \tag{3.11}$$

For $w, v \in H^1(\Omega)$, define

$$(w,v)_1 = \int_\Omega (w_x v_x + w_y v_y)\, dxdy. \tag{3.12}$$

Let $\eta \in H_0^2(\Omega) \cap H^3(\Omega), \eta_k \in V_k$ satisfy

$$\begin{cases} a(\eta, v) = (e_m - I_{k-1}^k\tilde{q}, v)_1, & \forall v \in H_0^2(\Omega), \\ b_k(\eta_k, v) = (e_m - I_{k-1}^k\tilde{q}, v)_1, & \forall v \in V_k. \end{cases}$$

Then we have

$$\begin{aligned} |e_m - I_{k-1}^k\tilde{q}|_1^2 &= b_k(\eta_k, e_m - I_{k-1}^k\tilde{q}) = b_k(\eta_k - I_{k-1}^k\Pi_{k-1}\eta, e_m - I_{k-1}^k\tilde{q}) \\ &\le C|\eta_k - I_{k-1}^k\Pi_{k-1}\eta|_{2,k}|e_m - I_{k-1}^k\tilde{q}|_{2,k} \\ &\le C(|\eta_k - \eta|_{2,k} + |\eta - I_{k-1}^k\Pi_{k-1}\eta|_{2,k})|e_m - I_{k-1}^k\tilde{q}|_{2,k}. \end{aligned}$$

From the following inequalities:

$$\begin{cases} |\eta|_{3,\Omega} \le C|e_m - I_{k-1}^k\tilde{q}|_1, \\ |\eta - \eta_k|_{2,k} \le Ch_k|e_m - I_{k-1}^k\tilde{q}|_1, \end{cases}$$

and inequalities (3.4), we get

$$|e_m - I_{k-1}^k\tilde{q}|_1^2 \le Ch_k|e_m - I_{k-1}^k\tilde{q}|_1|e_m - I_{k-1}^k\tilde{q}|_{2,k}.$$

It follows from (3.11) that

$$|e_m - I_{k-1}^k\tilde{q}|_{2,k} \le Ch_k|||e_m|||_{3,k}. \tag{3.13}$$

Let $f_k \in V_k, f_{k-1} \in V_{k-1}$ satisfy

$$(f_k, v)_1 = b_k(e_m, v), \quad \forall v \in V_k, \tag{3.14}$$

$$(f_{k-1}, v)_1 = b_k(e_m, I_{k-1}^k v), \quad \forall v \in V_{k-1}. \tag{3.15}$$

From inequalities (3.3) and (3.5), we have

$$|f_k|_1^2 \le |||e_m|||_{3,k}|||f_k|||_{1,k} \le C|||e_m|||_{3,k}|f_k|_1,$$

$$|f_{k-1}|_1^2 \le |||e_m|||_{3,k}|||I_{k-1}^k f_{k-1}|||_{1,k} \le C|||e_m|||_{3,k}|I_{k-1}^k f_{k-1}|_1 \le C|||e_m|||_{3,k}|f_{k-1}|_1.$$

Therefore,

$$|f_k|_1 \le C|||e_m|||_{3,k}, \qquad |f_{k-1}|_1 \le C|||e_m|||_{3,k}. \tag{3.16}$$

Let $\zeta \in H^3(\Omega) \cap H_0^2(\Omega)$ be determined by

$$a(\zeta, v) = (f_{k-1}, v)_1, \quad \forall v \in H_0^2(\Omega).$$

From (3.5) and (3.6) we have

$$b_{k-1}(q, v) = (f_{k-1}, v)_1, \quad \forall v \in V_{k-1}.$$

Then from the inequality

$$|\zeta|_3 \le C|f_{k-1}|_1$$

and (1.16), (3.16) and the inverse estimates, we have

$$|q|_{3,k} \le C|||e_m|||_{3,k}. \tag{3.17}$$

The following is obvious:

$$\begin{aligned}|I_{k-1}^k(q-\tilde q)|_{2,\Omega}^2 &\le b_k(I_{k-1}^k(q-\tilde q), I_{k-1}^k(q-\tilde q)) = b_k(I_{k-1}^k q - q, I_{k-1}^k(q-\tilde q)) \\ &\quad + b_k(q - I_{k-1}^k\tilde q, I_{k-1}^k(q-\tilde q)).\end{aligned} \tag{3.18}$$

From the Schwarz inequality and (3.3) and (3.17), we have

$$\begin{aligned}|b_k(I_{k-1}^k q - q, I_{k-1}^k(q-\tilde q))| &\le C|q - I_{k-1}^k q|_{2,k}|I_{k-1}^k(q-\tilde q)|_{2,k} \\ &\le Ch_k|||e_m|||_{3,k}|I_{k-1}^k(q-\tilde q)|_{2,k}.\end{aligned} \tag{3.19}$$

Let $\xi \in H_0^2(\Omega) \cap H^3(\Omega), \hat q \in V_{k-1}$ satisfy

$$a(\xi, v) = (f_k, v)_1, \quad \forall v \in H_0^2(\Omega), \tag{3.20}$$

$$b_{k-1}(\hat q, v) = (f_k, v)_1, \quad \forall v \in V_{k-1}. \tag{3.21}$$

Then

$$|\xi|_3 \le C|f_k|_1 \le C|||e_m|||_{3,k}, \tag{3.22}$$

$$|\xi - \hat q|_{2,k} \le Ch_k|f_k|_1. \tag{3.23}$$

Noticing that $I_{k-1}^k\tilde q$ is the finite element approximate solution to ξ in $I_{k-1}^k V_{k-1}$, by $I_{k-1}^k V_{k-1} \subset V_k$ and inequality (3.4), we have

$$|\xi - I_{k-1}^k\tilde q|_{2,k} \le Ch_k|\xi|_3. \tag{3.24}$$

By (3.3), (3.16) and (3.21) and inverse estimates for polynomials, we have

$$|\hat{q} - q|_{2,k-1}^2 \le b_{k-1}(\hat{q} - q, \hat{q} - q) = (f_k, \hat{q} - q - I_{k-1}^k(\hat{q} - q))_1$$
$$\le |f_k|_1 |\hat{q} - q - I_{k-1}^k(\hat{q} - q)|_1 \le Ch_k |f_k|_1 |\hat{q} - q|_{2,k-1}.$$

Therefore,

$$|\hat{q} - q|_{2,k-1} \le Ch_k |f_k|_1. \tag{3.25}$$

Combining (3.16) and (3.22) with (3.25), we have

$$|b_k(q - I_{k-1}^k \tilde{q}, I_{k-1}^k(\tilde{q} - q))| \le C|I_{k-1}^k \tilde{q} - q|_{2,k} |I_{k-1}^k(\tilde{q} - q)|_{2,k}$$
$$\le Ch_k |||e_m|||_{3,k} |I_{k-1}^k(\tilde{q} - q)|_{2,k}. \tag{3.26}$$

Noticing $\tilde{e}_{m+1} = e_m - I_{k-1}^k q = e_m - I_{k-1}^k \tilde{q} + I_{k-1}^k(\tilde{q} - q)$, we obtain (3.9) from (3.13), (3.18), (3.19) and (3.26).

Now we give convergence of the kth-level iteration. To simplify the notation, we define the following statement:

(S_k) When the kth-level iteration is applied to problem (P), we have

$$|z - MG(k, z_0, G)|_{2,k} \le \gamma |z - z_0|_{2,k}.$$

Lemma 6. *There exist $\gamma \in (0,1)$ and integer m large enough, both independent of the mesh parameter k, such that*

$$(S_{k-1}) \Longrightarrow (S_k).$$

Proof. From (3.6), (3.7) and (3.2), we have

$$|q|_{2,k} \le C|e_0|_{2,k}. \tag{3.27}$$

From (3.2), (3.8), (3.9), (3.27) and (S_{k-1}), we have

$$|z - MG(k, z_0, G)|_{2,k} \le |e_{m+1}|_{2,k} \le C|\tilde{e}_{m+1}|_{2,k} + |I_{k-1}^k(q - q_t)|_{2,k}$$
$$\le C(1+m)^{-1/4}|e_0|_{2,k} + C|q - q_t|_{2,k-1} \le C(1+m)^{-1/4}|e_0|_{2,k} + C\gamma^t |q|_{2,k-1}$$
$$\le (C(1+m)^{-1/4} + C\gamma^t)|e_0|_{2,k}.$$

If $\gamma \in (0,1)$ is small enough, then $C\gamma^t < \frac{\gamma}{2}$ (since $t > 1$). If m is large enough, then $C(m+1)^{-1/4} < \frac{\gamma}{2}$. For such choices we have

$$|z - MG(k, z_0, G)|_{2,k} \le \gamma |e_0|_{2,k}.$$

Since the first-level iteration is a direct method, the following theorem is a trivial consequence of Lemma 6.

Theorem 1. *If the number of smoothing steps m is large enough, then the kth-level iteration is a contraction for the norm $|\cdot|_{2,k}$. Moreover, the contraction number is independent of the mesh parameter k.*

Inequalities (1.15), (1.16), (3.2) and (3.4) imply the following lemma.

Lemma 7. *Let u_k be the solution of* (1.11). *Then*

$$|u_k - I_{k-1}^k u_{k-1}|_{2,k} \le Ch_k|u|_3 \tag{3.28}$$

where u is the solution of (1.4).

A standard argument yields the convergence of the full multigrid algorithm from the estimates in (3.28).

Theorem 2. *If the parameter r in the full multigrid algorithm is chosen large enough, then there exists a constant C such that*

$$|u - \tilde{u}_k|_{2,k} \le Ch_k|u|_3. \tag{3.29}$$

Finally, we note that since the number of nonzero entries in the stiffness matrices, the smoothing iteration matrices and the intergrid transfer matrices are all proportional to n_k, and since $t = 2$ or 3, the cost for obtaining $\tilde{u}_k$ is $\mathcal{O}(n_k)$. For the method of proof, we refer the reader to [3].

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