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ON THE PROBLEM OF BEST RATIONAL APPROXIMATION WITH INTERPOLATING CONSTRAINTS (II)* 1)

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Abstract

This paper is a continuation of [2]. In this part of the conjoint paper, we establish some results about uniqueness/non-uniqueness, the properties for the set of best approximants, strong uniqueness and continuity of the best approximation operator.

§4. Uniqueness

In [1], Wang has obtained the following important result: Let $R \in \mathbf{R}_1(m,n)$ be a best approximant of f in $\mathbf{R}_1(m,n)$. If $r(x) = f(x) - R(x)$ alternates at least $m+n-d(R)-k+2$ times according to $w(x)$, then the best approximant is unique.

Now we illustrate by an example that the best approximant may be unique as well even if the alternation condition above is not true.

Example 4. Take $m = 0$, $n = 1$, $\mathbf{R}_1(m,n)$ is the same as the one in Example 3. Let $f = x^2$. Then $R = 1 \in \mathbf{R}_1(m,n)$ is the unique best approximant of f in $\mathbf{R}_1(m,n)$. The uniqueness can be verified by computing the maximum value of the function

$$\frac{1}{ax+1} - x^2, \ x \in [-1,1], \ a \in (-1,1).$$

However, $R = 1$ does not have *deviation point* except a *neutral deviation point* $x = 0$. If we draw a curve of the discontinuous function $h(x) \equiv sign\ w(x)(f-R)(x)$, we can see that $h(x)$ has the alternating property. This fact indicates the existence of more general conditions under which the uniqueness is guaranteed. In order to find these conditions, we shall in this paper treat *neutral deviation points* as *non-neutral deviation points*.

Now we state a simple lemma.

Lemma 4. *Let $f \in \mathbf{C}[a,b] \setminus \mathbf{R}_1(m,n)$, $R \in \mathbf{R}_1(m,n)$, $r(x) = f(x) - R(x)$. Then for any $y \in Y(R)$,*

$$\liminf_{x \to y+} \frac{|r(x) - r(y)|}{|x-y|^\alpha} = 0, \ \alpha > o \ , \tag{4.1}$$

if and only if there do not exist $\lambda > 0$, $\eta > 0$, such that

$$0 < x - y < \eta,$$

$$|r(x)| + \lambda |x-y|^\alpha \le |r(y)|.$$

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For the left limit, a similar conclusion holds.

Let

$$X_1^{\pm}(R) = X_1(R) = \{x : x \in Y(R) \setminus X\},$$

$$X_2^{+}(R) = \{x \in Y(R) : \exists j \ s.t. \ x = x_j \in X, \ k_j \ is \ an \ odd \ integer, \\ \text{and } \liminf_{t \to x+} \frac{|r(t) - r(x)|}{(t-x)^{k_j+1}} = 0 \},$$

$$X_3^{+}(R) = \{x \in Y(R) : \ \exists j \ s.t. \ x = x_j \in X, \ k_j \ is \ an \ even \ integer, \\ \text{and } \liminf_{t \to x+} \frac{|r(t) - r(x)|}{(t-x)^{k_j+2}} = 0 \},$$

$$X_4^{+}(R) = \{x \in Y(R) \setminus X_3^{+}(R) : \ \exists j \ s.t. \ x = x_j \in X, \ k_j \ is \ an \ even \ integer, \\ \text{and } \liminf_{t \to x+} \frac{|r(t) - r(x)|}{(t-x)^{k_j+1}} = 0 \},$$

Similarly, $X_2^-(R), X_3^-(R)$ and $X_4^-(R)$ can be defined. For convenience of discussion, we assume next

$$X_i(R) \equiv X_i^+(R) = X_i^-(R), \quad i = 2, 3, 4.$$

We shall see that the points in $\bigcup_{i=1}^3 X_i^{\pm}(R)$ play the part of *deviation points* as ordinary best rational approximations.

Let $h(x) = sign \ w(x)r(x), \ a \le t_1 \le t_2 \le \cdots \le t_p \le b$. If

(i) $t_i \in \bigcup\limits_{j=1}^{3} X_j(R)$ for $i = 1, 2, ..., p$,

(ii) there exist u_s in any neighborhood of t_s such that

$$u_1 < u_2 < ... < u_p,$$

$$h(u_s)h(u_{s+1}) < 0, \quad for \ [t_s, t_{s+1}] \cap X_4(R) = \phi,$$

(iii) for any $\{\hat{t}_i\}_1^k$ that satisfies (i) and (ii), $k \le p$, then we say $r(t)$ is weakly alternate according to $w(t)$ on $\{t_i\}_1^p$, and $\{t_i\}_1^p$ is called a *weak alternant*.

Since u_s, which satisfies condition (ii), can be close to t_s sufficiently, $sign \ h(u_s) = 1 (\text{or } -1)$ is a constant. We denote it by $\alpha(t_s)$.

For $i = 1, ..., p-1$, let

$$l_i(R) = \begin{cases} 1, \text{ if } \ (-1)^{c_i(R)} h(t_i-)h(t_{i+1}+) < 0 \text{ and } t_i < t_{i+1}, \\ \quad c_i(R) \ne 0 \text{ (or } c_i(R) = 0, \text{ but } (t_i, t_{i+1}) \cap X_3(R) = \phi), \\ 0, \ \text{otherwise}, \end{cases}$$

where $c_i(R) = card\{[t_i, t_{i+1}] \cap X_4(R)\}, \ for \ t_i < t_{i+1}$.

It is easy to show that $l_i(R)$, $c_i(R)$ depend on R but are independent of t_i $(i = 1, 2, ..., p)$.

Now we can establish the unicity theorem.

Theorem 5. *Let $R \in \mathbf{R}_1(m,n)$ be a best approximation to $f \in C[a,b] \setminus \mathbf{R}_1(m,n)$ from $\mathbf{R}_1(m,n)$. Then the best approximation of f is unique if and only if*

$$m + n + 1 - d(R) - k \le c + \sum_{i=1}^{p-1} l_i(R) \ ,$$

where $c = \text{card}\{X_3(R) \cup X_4(R)\}$.

Proof . Sufficiency. If the unicity does not hold, there exists another best approximant $R_1 = P_1/Q_1 \in \mathbf{R}_1(m,n)$. Let $R = P/Q$. There is a polynomial $p_1 \in H_{m+n-d(R)-k}$ by Lemma 2, such that

$$R_1 - R = \frac{wp_1}{QQ1}. \tag{4.2}$$

Hence, if we put $r_1(x) = f(x) - R_1(x)$, one has

$$r(x) - r_1(x) = \frac{wp_1}{QQ1}, \tag{4.3}$$

$$\text{sign} w(x)(r(x) - r_1(x)) = \frac{|w(x)|p_1}{QQ1}. \tag{4.4}$$

Next we show that $p_1 = 0$. Let $\{t_i\}_1^p$ be the set of *weak alternant*.

(1) We prove first the following fact: There exists u_i in the neighborhood of t_i for $i = 1, 2, ..., p$, such that

$$u_1 \le u_2 \le ... \le u_p, \text{ and } u_i \ne u_{i+2}, \; for \; 1 \le i \le p-2, \tag{4.5}$$

$$\alpha(t_i)p_1(u_i) \ge 0, \quad i = 1, 2, ..., p, \tag{4.6}$$

and

$$p_1(u_i) = p_1'(u_i) = 0, \tag{4.7}$$

if $u_i = u_{i+1}$.

We prove the fact in three cases.

a) If there exist t_i, t_{i+1} $(1 \le i \le p-1)$ such that $t_i = t_{i+1}$, then $t_i \in X_3(R)$. Hence from (4.4) we have

$$\liminf_{t \to t_i+(\text{or}-)} \left| \frac{p_1(t)}{t - t_i} \right| = 0 \tag{4.8}$$

for $t_i \in X_3^+(R_1)$ $(or\ t_i \in X_3^-(R_1))$. Therefore (4.7) holds by taking $u_i = u_{i+1} = t_i$. If $t_i \notin X_3^+(R_1) \cup X_3^-(R_1)$, then by (4.4) there are u_i, u_{i+1} such that $u_i < t_i < u_{i+1}$, (4.6) holds and

$$max\{|t_i - u_i|, |t_i - u_{i+1}|\} < \frac{1}{2} min\{|t_i - t_{i-1}|, |t_i - t_{i+2}|\}.$$

b) If $t_{i-1} < t_i < t_{i+1}$ (or $t_i < t_{i+1}$ $(i = 1)$, or $t_{i-1} < t_i$ $(i = p)$), and $t_i \in X_2^+(R)(or\ X_2^-(R))$, then $p_1(t_i) = 0$ for $t_i \in X_2^+(R_1)(or\ X_2^-(R))$. Therefore (4.6) is valid by taking $u_i = t_i$. If $t_i \notin X_2^+(R_1)(or\ X_2^-(R))$, then there is u_i in the neighberhood of t_i such that (4.6) is true.

c) If $t_i \in X_1(R)$, then (4.6) is valid for $u_i = t_i$.

In addition, (4.5) is valid also according to the definitions of $u's$.

(2) Now we prove another fact:

$$p_1(u) = 0 \;, \forall u \in X_3(R) \cup X_4(R). \tag{4.9}$$

Indeed, if $u \in X_4^{\pm}(R_1)$, then we have (4.9) from (4.4). If $u \notin X_4^{\pm}(R_1)$, then there exist u', u'' in any neighborhood of u, such that

$$|r_1(u')| \le |r(u')|, \quad |r_1(u'')| \le |r(u'')|.$$

It follows that $p_1(u')p_1(u'') \leq 0$. Hence p_1 has a zero between u' and u''. But u' and u'' can close to u sufficiently, then $p_1(u) = 0$.

Now we can draw a conclusion from facts (1) and (2) that p_1 has at least $c + \sum_{i=1}^{p-1} l_i(R)$ zeros. Therefore $p_1 = 0$ by using the fact that $p_1 \in \mathbf{H}_{m+n-d(R)-k}$, and hence the unicity is proved.

The proof of necessity is left to the next section.

Corollary 4. If $f(x) = g(x)$, $\forall x \in X$, then the existence of the best approximation to f from $\mathbf{R}_1(m,n)$ implies the unicity.

Proof . We may assume $f \notin \mathbf{R}_1(m,n)$. Let R be a best approximation to f from $\mathbf{R}_1(m,n)$. Then $Y(R) \cap X = \phi$; hence $X_i^{\pm}(R) = \phi$ *for* $i = 2, 3, 4$. It follows from Theorem 4 and Theorem 5 that the best approximation is unique.

§5 The Set of Best Approximants

Let R be a best approximation to $f \in \mathbf{C}[a,b]$ from $\mathbf{R}_1(m,n)$. From Theorem 5 it follows that, if $m + n - d(R) \not- k + 1 > c + \sum_{i=1}^{p-1} l_i(R)$, then the best approximants are not unique. Define

$$\Re(f) = \{R \in \mathbf{R}_1(m,n) : \ \|f - R\| = \inf_{T \in \mathbf{R}_1(m,n)} \|f - T\|\},$$

$$\mathbf{P}_1 - R\mathbf{Q}_1 = \{P_1 - RQ_1 : P_1/Q_1 \in \Re(f)\}.$$

Now we consider the characterization of the class $\mathbf{P}_1 - R\mathbf{Q}_1$. First of all, one has

Lemma 5. (i) *The class* $\mathbf{P}_1 - R\mathbf{Q}_1$ *is a convex set containing the origin.*
(ii) *Let* $P_i - RQ_i \in \mathbf{P}_1 - R\mathbf{Q}_1$, $i = 1, 2$. *If*

$$P_1 - RQ_1 \neq C(P_2 - RQ_2) \quad (C \textit{ is a constant}),$$

then

$$\frac{P_1}{Q_1} \neq \frac{P_2}{Q_2}.$$

Proof. Let $R_i = P_i/Q_i \in \Re(f)$ for $i = 0, 1$. We shall show

$$R_\lambda = \frac{\lambda P_0 + (1-\lambda)P_1}{\lambda Q_0 + (1-\lambda)Q_1} \in \Re(f), \ \forall \lambda \in [0,1].$$

From the lemma in [3], it follows that $R_\lambda \in \mathbf{R}_1(m,n)$. For a fixed $x \in [a,b]$, considering $R_\lambda(x)$ as a function of λ, we can show easily that $dR_\lambda(x)/d\lambda$ does not change sign. Hence

$$\min\{R_0(x), R_1(x)\} \leq R_\lambda(x) \leq \max\{R_0(x), R_1(x)\}.$$

We get at once from the above inequalities that $R_\lambda \in \Re(f)$. Therefore $\mathbf{P}_1 - R\mathbf{Q}_1$ is a convex set. It is obvious that $0 \in \mathbf{P}_1 - R\mathbf{Q}_1$. Consequently, assertion (i) is proved. The proof of assertion (ii) is easy.

Let

$$\ell(\mathbf{P}_1 - R\mathbf{Q}_1) = \{\lambda f_1 + (1-\lambda) f_2 : f_1, f_2 \in \mathbf{P}_1 - R\mathbf{Q}_1, \lambda \in \mathbf{R}\},$$

$$\dim(\mathbf{P}_1 - R\mathbf{Q}_1) \equiv \dim(\ell(\mathbf{P}_1 - R\mathbf{Q}_1)),$$

where *dim* denotes dimension. Since $\mathbf{P}_1 - R\mathbf{Q}_1$ contains origin, $\ell(\mathbf{P}_1 - R\mathbf{Q}_1)$ is a finite dimensional linear space.

Now we attempt to establish a relation between $dim(\mathbf{P}_1 - R\mathbf{Q}_1)$ and the weak alternant of R. To this end we introduce the following

Lemma 6. *Let A be a convex set and $0 \in A$. Then $dimA$ is equal to the number of linearly independent elements contained in A.*

Theorem 6. *Let $R = P/Q \in \Re(f)$, $m+n+1-d(R)-k > c + \sum_{i=1}^{p-1} l_i(R)$. Then*

$$\dim(\mathbf{P}_1 - R\mathbf{Q}_1) = m+n+1-d(R)-k-c,$$

where c is independent of $R \in \Re(f)$.

Proof. a) To begin with, we shall show

$$\dim(\mathbf{P}_1 - R\mathbf{Q}_1) \geq m+n+1-d(R)-k-c.$$

Take

$$b_i \in [t_i, t_{i+1}] \setminus Y(R), \quad for\ l_i(R) = 1,\ i = 1, 2, ..., p-1.$$

It can be seen clearly that b_i can vary arbitrarily in an interval. Let

$$\Omega(x) = \theta \prod (x - b_i) \prod_{z_j \in X_3(R) \cup X_4(R)} (x - z_j) h(x) = \theta \prod (x - b_i) F(x) h(x),$$

where $\theta = \pm 1$ is a constant, and $h(x) \in \mathbf{H}_t$ has no zeors on $[a, b]$, where $t = m+n-d(R)-k-c-\sum_{i=1}^{p-1} l_i(R)$.

By the definition of b_i, z_j, one has

(i) $\Omega \in \mathbf{H}_{m+n-d(R)-k}$,

(ii) θ can be chosen so that $\Omega(x)$ and $w(x)r(x)$ have the same sign on $\{t_i\}$.

From Lemma 2, it follows that there exists $(P_1, Q_1) \in \mathbf{L}(m, n)$, such that

$$P_1 - RQ_1 = \frac{w\Omega}{Q}.$$

We shall show in what follows that, if λ (> 0) is small enough,

$$R_\lambda = \frac{P + \lambda P_1}{Q + \lambda Q_1} \in \Re(f).$$

First

$$r_\lambda(x) = f(x) - R_\lambda(x) = r(x) - \lambda \frac{w(x)\Omega(x)}{Q(x)(Q(x) + \lambda Q_1(x))}. \tag{5.1}$$

Since for any $x_i \in Y(R) \setminus \bigcup_{i=1}^{3} X_i(R)$ we have

$$\liminf_{t\to x_i} \frac{|r(t)-r(x_i)|}{|t-x_i|^{k_i+2}} \neq 0, \quad x_i \in X_4(R),$$

or

$$\liminf_{t\to x_i} \frac{|r(t)-r(x_i)|}{|t-x_i|^{k_i+1}} \neq 0, \quad x_i \in Y(R) \setminus \bigcup_{i=1}^{4} X_i(R),$$

from Lemma 4 there exist $\eta_i > 0$, $\hat{\lambda}_i > 0$, such that

$$|r(t)| + \hat{\lambda}_i |t-x_i|^{k_i+2} \le |r(x_i)|, \quad x_i \in X_4(R),$$

or

$$|r(t)| + \hat{\lambda}_i |t-x_i|^{k_i+1} \le |r(x_i)|, \quad x_i \in Y(R) \setminus \bigcup_{i=1}^{4} X_i(R),$$

if $t \in E_i \equiv \{t : |x_i - t| \le \eta_i\}$. Therefore there exists $\lambda_i > 0$, such that

$$|r(t)| + \lambda_i \left| \frac{w(t)\Omega(t)}{Q(t)(Q(t)+\lambda_i Q_1(t))} \right| \le |r(x_i)|, \quad \forall t \in E_i. \tag{5.2}$$

Let

$$\Phi = \{y \in [a,b] : w(y)r(y)\Omega(y) \ge 0\}, \quad \Psi = \bigcup_{x_i \in Y(R)\setminus \cup_{i=1}^{3} X_i(R)} E_i,$$

$$\Theta = [a,b] \setminus (\Phi \cup \Psi), \quad \mu = \sup_{x\in\Theta} |r(x)|.$$

Then we can prove $\mu < \|r\|$. For the converse, there exist $y_i \longrightarrow y$, such that

$$y_i \in \Theta, \quad |r(y_i)| \longrightarrow |r(y)| = \|r\|.$$

We may assume $y_i > y$. If $y \in \cup_{i=1}^{3} X_i(R)$, then $w(y_i)r(y_i)\Omega(y_i) > 0$ for i large enough. Hence $y_i \in \Phi$. This is a contradiction. If $y_i \in Y(R) \setminus \cup_{i=1}^{3} X_i(R)$, then $y_i \in E_i \subset \Psi$ when i sufficiently large. This contradicts $y_i \in \Theta$. Therefore $\mu < \|r\|$. Take λ, such that

$$\|R_\lambda - R\| \le \|r\| - \mu, \tag{5.3}$$

$$0 < \lambda \le \min\{\lambda_i : x_i \in Y(R) \setminus \cup_{i=1}^{3} X_i(R)\}.$$

Then it follows from (5.1) and (5.3) that

$$|r_\lambda(x)| \le \|r\|, \quad \forall x \in \Phi \,. \tag{5.4}$$

If $x \in \Psi$, (5.2) implies (5.4). If $x \in \Theta$,

$$|r_\lambda(x)| \le |r(x)| + \|R_\lambda - R\| \le \sup_{x\in\Theta} |r(x)| + \|r\| - \mu = \|r\|.$$

In a word, (5.4) holds for all $x \in [a,b]$. Therefore $r_\lambda(x) \in \Re(f)$.

Since

$$(P+\lambda P_1) - (Q+\lambda Q_1)R = \lambda(P_1 - Q_1 R) = \lambda \frac{w\Omega}{Q} \in \mathbf{P}_1 - R\mathbf{Q}_1 \,,$$

and b_i's can vary in an interval and h is any polynomial of degree $m+n-d(R)-k-c-\sum_{i=1}^{p-1} l_i(R)$ without zero on $[a,b]$ according to the construction of $\Omega(x)$, we can choose $m+n-d(R)-k-c+1$ linearly independent functions from the class $\{\Omega(x)\}$. Then by Lemma 6 we have

$$\dim(\mathbf{P}_1 - R\mathbf{Q}_1) \geq m+n-d(R)-k-c+1\,. \tag{5.5}$$

b) Now we show that (5.5) is an equality. To this end we need only to prove that for any best approximant $R_1 = P_1/Q_1$, if we put

$$P_1 - RQ_1 = \frac{wp}{Q}, \quad p \in \mathbf{H}_{m+n-d(R)-k},$$

then p can be expressed linearly by $\Omega's$.

Similarly to the proof of (4.9), we can deduce that for any $x \in X_3(R) \cup X_4(R)$, $p(x) = 0$. That is,

$$p(x) = \Big(\prod_{x_j \in X_3(R)\cup X_4(R)} (x - z_i)\Big) p_1(x) = F(x)p_1(x).$$

Then p can be expressed by the linear combination of $\Omega's$. Therefore (5.5) is equality.

We prove now that c is independent of $R \in \Re(f)$. That is, we need to show the set $X_3(R) \cup X_4(R)$ is independent of $R \in \Re(f)$. For this purpose, we shall show that for any $t = x_j \in X$,

$$\liminf_{x\to t} \left|\frac{r_1(x) - r_1(t)}{(x-t)^{k_j+1}}\right| = 0\,, r_1(x) = f(x) - R_1(x)\,, \tag{5.6}$$

so long as

$$\liminf_{x\to t} \left|\frac{r(x) - r(t)}{(x-t)^{k_j+1}}\right| = 0\,, r(x) = f(x) - R(x)\,.$$

In fact, from (4.3) we have

$$r(x) - r_1(x) = \frac{w(x)p(x)}{Q(x)Q1(x)},$$

and $p(x_j) = 0$ by (4.9). Hence

$$\begin{aligned} \left|\frac{r_1(x) - r_1(t)}{(x-t)^{k_j+1}}\right| &= \left|\frac{r_1(x) - r(x) + r(x) - r_1(t)}{(x-t)^{k_j+1}}\right| \\ &\leq \left|\frac{r_1(x) - r(x)}{(x-t)^{k_j+1}}\right| + \left|\frac{r(x) - r_1(t)}{(x-t)^{k_j+1}}\right| = \left|\frac{w(x)p(x)}{Q(x)Q_1(x)(x-t)^{k_j+1}}\right| + \left|\frac{r(x) - r(t)}{(x-t)^{k_j+1}}\right|. \end{aligned}$$

Therefore

$$\begin{aligned} &\liminf\nolimits_{x\to t} \left|\frac{r_1(x) - r_1(t)}{(x-t)^{k_j+1}}\right| \\ &\leq \liminf\nolimits_{x\to t} \left|\frac{w(x)p(x)}{Q(x)Q_1(x)(x-t)^{k_j+1}}\right| + \liminf\nolimits_{x\to t} \left|\frac{r(x) - r(t)}{(x-t)^{k_j+1}}\right| = 0, \end{aligned}$$

i.e., (5.6) holds. The theorem is proved.

Now we can finish the necessity proof of Theorem 5. Suppose the best approximant is unique. We assert the fact that $m+n-d(R)-k+1 \le c+\sum_{i=1}^{p-1} l_i(R)$. Otherwise, Theorem 6 leads to a contradiction.

§6. Strong Unicity

Let $R^* \in \Re(f)$. If for any $R \in \mathbf{R}_1(m,n)$

$$\|f-R\| \ge \|f-R^*\| + \gamma\|R-R^*\|, \tag{6.1}$$

where $\gamma(>0)$ is independent of R, then R^* is said to be *strong unique*.

In order to establish the conditions under which strong unicity is guaranteed, we quote the following

Lemma 7. *Let $R \in \mathbf{R}_1(m,n)$ be a best approximation to $f \in \mathbf{C}[a,b] \setminus \mathbf{R}_1(m,n)$ from $\mathbf{R}_1(m,n)$. If*

(i) $m+n<k+d(R)$ or

(ii) $m+n \ge k+d(R)$ but $r(x)=f(x)-R(x)$ alternates at least $m+n-d(R)-k+2$ times according to $w(x)$, then for any $\phi \in \mathbf{P}-R\mathbf{Q}$, $\phi \ne 0$,

$$\inf_{y\in Y(R)} \phi(y)(f-R)(y) < 0. \tag{6.2}$$

Proof. The case $m+n<d(R)+k$ is trivial. In what follows we assume $m+n \ge d(R)+k$. If (6.2) is not true, there exists $\phi = P_0 - RQ_0 \in \mathbf{P}-R\mathbf{Q}$, $\phi \ne 0$, such that

$$\phi(y)(f-R)(y) \ge 0,\ y \in Y(R).$$

From Lemma 2, there is $p \in \mathbf{H}_{m+n-d(R)-k}$, such that

$$w(y)(f-R)(y)p(y) \ge 0,\ y \in Y(R).$$

Since $w(y)(f-R)(y)$ alternates at least $m+n-d(R)-k+2$ times, there exist t_i 's such that

$$a \le t_0 < t_1 < \dots < t_l \le b \quad (l \ge m+n-d(R)-k+1)$$

and (3.5) holds. Hence $p=0$, i.e. $\phi=0$, a contradiction.

Referring to the proof of strong unicity theorem for the ordinary best rational approximation (see [4],p.165), using Lemma 7 we can establish the following theorem.

Theorem 7. *Let $R \in \mathbf{R}_1(m,n)$ be a best approximation to $f \in \mathbf{C}[a,b] \setminus \mathbf{R}_1(m,n)$ from $\mathbf{R}_1(m,n)$, and $d(R)=0$. Then R is strong unique provided that the conditions of Lemma 7 are satisfied.*

The proof of the theorem is omitted.

It should be pointed out that the condition that $r(x)$ alternates at least $m+n-d(R)-k+2$ times according to $w(x)$ can not be left out in the theorem. Namely, the unicity and $d(R)=0$ do not imply strong unicity. To illustrate the point, we give an example.

Example 5. Suppose the problem considered here is the same as the one in Example 4. Let $a_k = 1/k$, $R_k = 1/(a_k x + 1)$. We shall show that

$$\lim_{k\to\infty} \frac{\|f - R_k\| - \|f - R\|}{\|R_k - R\|} = 0, \tag{6.3}$$

where $R = 1$ is the best approximation to $f = x^2$ from $\mathbf{R}_1(m,n)$. Since

$$f'(x) - R_k'(x) = 2x + \frac{k}{(x+k)^2} = \frac{2x(x+k)^2 + k}{(x+k)^2},$$

the equation $f'(x) - R_k'(x) = 0$ has only one root in the interval $[-1,1]$. Let x_k be such a root. Then

$$\left|x_k + \frac{1}{2k}\right| = \left|\frac{1}{2k} - \frac{k}{2(x_k+k)^2}\right| < \frac{1}{(k-1)^2}, \quad k > 1,$$

$$x_k = -\frac{1}{2k} + \frac{\theta_k}{(k-1)^2} = -0.5k^{-1} + \theta_k(k-1)^{-2}, \quad |\theta_k| < 1.$$

It follows that

$$\|f - R_k\| = |f(x_k) - R_k(x_k)| = 1 + \frac{1 - 2k\theta_k(k-1)^{-2}}{2k^2 + 2\theta_k k(k-1)^{-2} - 1} - (0.5k^{-1} - \theta_k(k-1)^{-2})^2.$$

In addition,

$$\|f - R\| = 1, \ \|R_k - R\| = \frac{1}{k-1}.$$

Hence

$$\frac{\|f - R_k\| - \|f - R\|}{\|R_k - R\|} = O(k^{-1}).$$

Therefore (6.3) holds, i.e., we do not have strong unicity.

§7. Continuity of the Best Approximation Operator

Denote

$$T: \ \mathbf{C}[a,b] \longrightarrow \mathbf{R}_1(m,n)\ ,$$

the best approximation operator, i.e., Tf is the best approximations to $f \in \mathbf{C}[a,b]$ from $\mathbf{R}_1(m,n)$. It is obvious from the discussions above that T may not be well defined and may have many values. However, we have the following statement.

Theorem 8. *Let $R_0 \in \mathbf{R}_1(m,n)$ be a best approximation to $f_0 \in \mathbf{C}[a,b] \setminus \mathbf{R}_1(m,n)$ from $\mathbf{R}_1(m,n)$. Assume $d(R_0) = 0$, and R_0 has no neutral deviation point. Then there exists $\varepsilon > 0$, such that*

(i) *Tf is defined uniquely.*

(ii) *$\|Tf - Tf_0\| \le \beta\|f - f_0\|$, $\beta > 0$ is a constant independent of f,*

provided that $\|f - f_0\| < \varepsilon$, $f \in \mathbf{C}[a,b]$.

Proof. Under the assumptions of the theorem, strong unicity holds for f_0, namely

$$\|f_0 - R\| \ge \|f_0 - R_0\| + \gamma\|R - R_0\|, \gamma > 0\ . \tag{7.1}$$

For a given $f \in C[a,b]$, define

$$S_1(f) = \{R \in \mathbf{R}_1(m,n) : \|R - f\| \le \|R_0 - f\|\},$$
$$S_2(f) = \{R \in \mathbf{R}_1(m,n) : \|R - R_0\| \le 2\gamma^{-1}\|f - f_0\|\}.$$

Then for $R \in S_1(f)$, from (7.1), we have

$$\begin{aligned}\gamma\|R - R_0\| &\le \|f_0 - R\| - \|f_0 - R_0\| \le \|f_0 - f\| + \|f - R\| - \|f_0 - R_0\| \\ &\le \|f_0 - f\| + \|f - R_0\| - \|f_0 - R_0\| \le 2\|f_0 - f\|.\end{aligned}$$

Hence $R \in S_2(f)$, and then

$$S_1(f) \subset S_2(f)\ ,$$
$$d_1(f) = \inf_{R \in S_1(f)} \|R - f\| = \inf_{R \in S_2(f)} \|R - f\|. \tag{7.2}$$

We normalize the elements $P/Q,\ P_0/Q_0 \in S_2(f)$ such that $\|Q\| = \|Q_0\| = 1$. Let

$$\inf_{x \in [a,b]} Q_0(x) = 2\delta > 0\ .$$

Then we can prove that there exists $\varepsilon_1 > 0$, if $\|f_0 - f\| < \varepsilon_1$,

$$\|Q - Q_0\| < \delta, \tag{7.3}$$

for any $R = P/Q \in S_2(f)$. On the contrary, there exist $R_k = P_k/Q_k \in S_2(f_k), f_k \in C[a,b]$, such that

$$\|f_k - f_0\| \longrightarrow 0,\ \|P_k/Q_k - P_0/Q_0\| \longrightarrow 0,\ \|Q_k - Q_0\| \ge \delta.$$

We may assume $P_k \longrightarrow P$, $Q_k \longrightarrow Q$. By taking limit from the two sides of the equality $Q_k R_k = P_k$, we have $QR_0 = P$. From $d(R_0) = 0$ we get at once that $P = P_0, Q = Q_0$. This contradicts $\|Q_k - Q_0\| \ge \delta$. Therefore (7.3) holds.

From (7.3) it follows that for any $R = P/Q \in S_2(f)$,

$$\inf_{x \in [a,b]} Q(x) \ge \inf_{x \in [a,b]} Q_0(x) - \|Q - Q_0\| > \delta, \tag{7.4}$$

while from (7.2), there exists a sequence $R_k = P_k/Q_k \in S_2(f)$, such that

$$P_k \longrightarrow P^*,\ Q_k \longrightarrow Q^*, \quad \|R_k - f\| \longrightarrow d_1(f).$$

Then by (7.4) we have $\inf_{x \in [a,b]} Q^*(x) \ge \delta$. Therefore $P_k/Q_k \longrightarrow P^*/Q^*,\ P^*/Q^* \in S_2(f)$. That is, P^*/Q^* is a best approximation to f, and conclusion (ii) holds by taking $\beta = 2\gamma^{-1}$.

Next we prove the best approximation to f is unique. Let

$$\delta_1 \equiv \|f_0 - R_0\| - \max_{0 \le i \le s} |f_0(x_i) - R_0(x_i)|. \tag{7.5}$$

Then $\delta_1 > 0$ by the hypothesis of the theorem. Let R^* be a best approximation to f. Then

$$\begin{aligned}\max_{0 \le i \le s} |f(x_i) - R^*(x_i)| &\le \max_{0 \le i \le s} |f(x_i) - f_0(x_i)| + \max_{0 \le i \le s} |f_0(x_i) - R^*(x_i)| \\ &\le \|f - f_0\| + \max_{0 \le i \le s} |f_0(x_i) - R_0(x_i)| + \max_{0 \le i \le s} |R_0(x_i) - R^*(x_i)|.\end{aligned}$$

Using (7.5) and the equalities $R_0(x_i) = R^*(x_i)$, we have

$$\begin{aligned}\max_{0\le i\le s} |f(x_i) - R^*(x_i)| &\le \|f - f_0\| + \|f_0 - R_0\| - \delta_1 \\ &\le \|f - f_0\| + \|f - f_0\| + \|f - R_0\| - \delta_1 \\ &\le 2\|f - f_0\| + \|f - R^*\| + \|R^* - R_0\| - \delta_1.\end{aligned}$$

Since $R^* \in S_2(f)$, one has

$$\max_{0\le i\le s} |f(x_i) - R^*(x_i)| \le (2 + 2\gamma^{-1})\|f - f_0\| + \|f - R^*\| - \delta_1.$$

Take $\varepsilon \in (0, \varepsilon_1]$ such that, if $\|f - f_0\| < \varepsilon$,

$$(2 + 2\gamma^{-1})\|f - f_0\| - \delta_1 < 0 .$$

Hence

$$\max_{0\le i\le s} |f(x_i) - R^*(x_i)| < \|f - R^*\|,$$

i.e., $f - R^*$ does not have neutral deviation point. From Theorem 4 and Theorem 5 it follows that R^* is the unique approximation to f. Thus the proof of the theorem is completed.

Finally, we may see from the proof given above that conclusion (ii) is also true under the conditions of Theorem 7. But Tf may have many values in this case. Hence Tf in conclusion (ii) can be any best approximation to f.

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