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ON AN UNCONDITIONALLY STABLE SCHEME FOR THE UNSTEADY NAVIER–STOKES EQUATIONS*

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Abstract

Theoretical time step constraints of semi-implicit schemes are known to be more restrictive than should be in practice. We intend to alleviate the constraints with more smoothness assumptions on the solutions. By introducing a new scheme with modification on the treatment of the nonlinear term, we are able to prove that the scheme is unconditionally stable and convergent. Further more, we show that the modified scheme and the original semi-implicit one are equivalent under a weak condition on the time step and the number of space discretization points.

§1. Introduction

In numerical simulations of incompressible flow represented by the Navier-Stokes equations (1.1), one of the major difficulties is to construct a suitable time discretization scheme. The origin of such difficulty consists essentially of two parts:

(i) The pressure and the velocity in Navier-Stokes equations are coupled by the incompressibility constraint (1.1b) such that a direct inversion of the resulting discrete system is very expensive. A great number of fast Stokes solvers have been developed by using either an iterative method or a Green's function method (also called influence matrix method, see for instance [8]). Another remedy for removing this difficulty is to use the so called projection method initially proposed by A.J.Chorin and R.Temam (cf. [4], [11]) which separates the calculation of the pressure from that of the velocity. However, this kind of splitting schemes suffers from a large time splitting error which can only be removed by a sophisticated extrapolation process (cf. [10]).

(ii) The treatment of the nonlinear term: usually, explicit treatment of the nonlinear term leads to in some cases a restrictive theoretical time step constraint (see for instance [12]) while implicit treatment makes the resulting discrete system very difficult to be resolved.

In this paper, we concentrate on improving existing theoretical stability constraints for semi-implicit schemes in which the diffusion term is treated implicitly, leaving the convection term (i.e. nonlinear term) treated explicitly.

In many cases, one observes that a semi-implicit scheme gives stable results under a time step constraint which is much weaker than what the theoretical results predict, especially in cases where a smooth solution exists. A natural question one can ask is: can we improve the existing stability conditions by giving more smoothness assumptions on the solutions?

We will give a positive answer to this question by considering a concrete space discretization, namely, the Chebyshev-Galerkin approximation (we refer to [7] for a detailed presentation of this method). For other space discretizations, similar results could be obtained by

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using the same technique. The idea used here can also be applied to other time-dependent elliptic nonlinear systems.

The unsteady Navier-Stokes equations in the primitive variable formulation are written as

$$\frac{\partial u}{\partial t} - \nu\Delta u + u\nabla u + \nabla p = f(x,t)\ ,\quad (x,t) \in Q = \Omega \times [0,T], \tag{1.1a}$$

$$\text{div } u = 0\ ,\quad \text{in } Q, \tag{1.1b}$$

$$u(x,0) = u_0(x), \tag{1.1c}$$

$$u(x,t) = 0|_{\partial\Omega}\quad \forall t \in [0,T], \tag{1.1d}$$

where Ω is an open set in R^d $(d = 2\ or\ 3)$ with sufficiently smooth boundary, the unknowns are the vector function u (velocity) and the scalar function p (pressure). For the sake of simplicity, we assume that the velocity satisfies the homogeneous boundary condition.

We will restrict ourselves to the two dimensional case. More specifically, we consider $\Omega = (-1,1) \times (-1,1)$. The Chebyshev weight function defined in Ω is

$$\omega(x) = (1 - x_1^2)^{-\frac{1}{2}}(1 - x_2^2)^{-\frac{1}{2}} \quad \text{for}\ \ x = (x_1, x_2) \in \Omega.$$

The following functional spaces will be used in the sequel:

$$\mathcal{X} = \mathcal{H}^1_{0,\omega}(\Omega),$$
$$\mathcal{H}_\omega = \{u \in \mathcal{L}^2_\omega(\Omega) : \quad \text{div } u = 0,\quad u \cdot \vec{n} = 0\},$$
$$\mathcal{V}_\omega = \{u \in \mathcal{X} : \quad \text{div } u = 0\},$$
$$\tilde{\mathcal{V}}_\omega = \{u \in \mathcal{X} : \quad \text{div } (u \cdot \omega) = 0\},$$

where ω is the Chebyshev weight function, and $\mathcal{L}^2_\omega(\Omega)$ and $\mathcal{H}^1_{0,\omega}(\Omega)$ are weighted Sobolev spaces. To alleviate notations, we use calligraphic letters to denote vector function spaces, for instance, $\mathcal{L}^2_\omega = (L^2_\omega)^2$.

With the help of these functional spaces, we can reformulate the problem (1.1) as

$$\begin{cases} \text{find}\ \ u(t) \in \mathcal{V}_\omega\ \ \text{such that} \\ \dfrac{\partial}{\partial t}t(u,v)_\omega + \nu a_\omega(u,v) + (B(u),v)_\omega =< f, v >_\omega\ ,\quad \forall v \in \tilde{\mathcal{V}}_\omega, \\ u(0) = u_0 \end{cases} \tag{1.2}$$

where we have

$$(u,v)_\omega = (u, v\omega) = \int_\Omega uv\omega dx\ ,\ a_\omega(u,v) = (\nabla u, \nabla(v \cdot \omega)),$$

$$B(u) = \sum_{i=1}^{d} u_i \frac{\partial u}{\partial x_i} \ \text{and}\ < \cdot,\cdot >_\omega \ \text{the duality relation between}\ \mathcal{X}'\ \text{and}\ \mathcal{X}.$$

Due to the Chebyshev weight function involved here, the formulation (1.2) is not symmetric such that the existence of solutions for (1.2) is not covered by the conventional theory.

It is proved in [5] and [9] that the bilinear form $a_\omega(\cdot,\cdot)$ is continuous and coercive on $\mathcal{X} \times \mathcal{X}$. More precisely, we have

$$\left\{ \begin{array}{l} a_\omega(u,v) \le \alpha||u||_{1,\omega}||v||_{1,\omega} \ , \quad \forall u,v \in \mathcal{X}, \\ a_\omega(u,u) \ge \beta||u||^2_{1,\omega} \ , \quad \forall u \in \mathcal{X}. \end{array} \right. \tag{1.3}$$

Therefore, $||\cdot||_\omega = a_\omega(\cdot,\cdot)^{\frac{1}{2}}$ is a norm equivalent to $||\cdot||_{1,\omega}$ on $\mathcal{X}$. We will use in the sequel the norm $||\cdot||_\omega$ instead of $||\cdot||_{1,\omega}$ and we will denote the norm on $\mathcal{L}^2_\omega(\Omega)$ by $|\cdot|_\omega$.

In order to formulate a discrete Chebyshev-Galerkin approximation of (1.2), we define

S_N: The set of polynomials such that the order of each variable is less than or equal to N;

$$\mathcal{X}_N = S_N \cap \mathcal{X}, \mathcal{V}_N = S_N \cap \mathcal{V}_\omega \text{ and } \tilde{\mathcal{V}}_N = S_N \cap \tilde{\mathcal{V}}_\omega.$$

Let us consider first the following fully discretized scheme which consists of the second order Crank-Nicolson and Adams-Bashforth scheme in time and the Chebyshev-Galerkin scheme in space:

$$\left\{ \begin{array}{l} \text{Let } \ u_N^0 = \Pi_N u_0, \quad \text{find } \ u_N^n \in \mathcal{V}_N \ \text{ such that} \\ \frac{1}{k}(u_N^{n+1} - u_N^n, v)_\omega + \frac{\nu}{2} a_\omega(u_N^{n+1} + u_N^n, v) = < f^{n+\frac{1}{2}}, v >_\omega \\ \qquad -1.5(B(u_N^n), v)_\omega + 0.5(B(u_N^{n-1}), v)_\omega \ , \quad \forall v \in \tilde{\mathcal{V}}_N \end{array} \right. \tag{1.4}$$

where $k = \frac{T}{K}$: time discretization step and $f^{n+\frac{1}{2}} = \frac{1}{k}\int_{nk}^{(n+1)k} f(x,t)dt$;

Π_N : a projection operator $\mathcal{X} \to \mathcal{X}_N$ such that

$$a_\omega(u - \Pi_N u, v) = 0 \ , \quad \forall v \in \mathcal{X}_N \ , \ u \in \mathcal{X}. \tag{1.5}$$

Throughout the paper, we will use c and c_i to denote constants which can vary from one equation to another.

We infer from (1.5) and (1.3) that

$$a_\omega(u - \Pi_N u, u - \Pi_N u) = a_\omega(u - \Pi_N u, u - \phi) \le c||u - \Pi_N u||_\omega||u - \phi||_\omega,$$
$$\forall u \in \mathcal{X} \quad , \ \forall \ \phi \in \mathcal{X}_N. \tag{1.6}$$

Since the following is true (cf. [9]):

$$\inf_{\phi \in \mathcal{X}_N} ||u - \phi||_\omega \le cN^{1-s}||u||_{s,\omega} \ , \ \forall \ u \in \mathcal{H}^s_\omega(\Omega) \cap \mathcal{X}$$

we deduce from (1.3) and (1.6) that

$$||u - \Pi_N u||_\omega \le cN^{1-s}||u||_{s,\omega} \ , \ \forall \ u \in \mathcal{H}^s_\omega(\Omega) \cap \mathcal{X}. \tag{1.7}$$

R.Temam analyzed in [12] this kind of scheme in a general space discretization form. He introduced two quantities $S(N)$ and $S_1(N)$ defined by

$$\left\{ \begin{array}{l} ||u_N|| \le S(N)|u_N| \ , \quad \forall u_N \in S_N, \\ |((u_N \cdot \nabla)v_N, w_N)| \le S_1(N)|u_N| \cdot ||v_N|| \cdot |w_N| \ , \quad \forall u_N, v_N, w_N \in \mathcal{X}_N \end{array} \right. \tag{1.8}$$

and proved that the semi-implicit schemes are stable and convergent under the conditions:

$$kS^2(N) \leq C_1 \quad , \quad kS_1^2(N) \leq C_2. \tag{1.9}$$

These conditions are somewhat restrictive especially when a spectral method is adapted to the space discretization. For instance, we have $S(N) = O(N^2)$ for the Chebyshev or Legendre approximations (cf. [6]). It means that we should have a time step at least less than cN^{-4} to ensure the stability. This is evidently not a reasonable constraint in practice. It is then necessary to lighten this constraint by assuming more smoothness of the solution.

In the next section, we will introduce a modified scheme which differs from (1.4) in the treatment of the nonlinear term. We will prove the new scheme is unconditionally stable provided that the solution u is uniformly bounded in Q. Then, we will prove in Section 3 that the new scheme preserves the second order accuracy in time and the spectral accuracy in space. Finally, we will prove in Section 4 that the modified scheme and the original one are equivalent under a very weak condition.

§2. A Modified Scheme and Related Stability

We assume that u is uniformly bounded in Q, i.e.

(H1) There exists $M > 0$ such that $|u|_{L^\infty(Q)} \leq M$.

Following an idea of Bressan and Quarteroni (cf. [2]), we introduce the truncated function H: $R \to R$ defined as

$$H(y) = \begin{cases} y\,, & \text{if } |y| \leq 2M, \\ 2M\,, & \text{if } y \geq 2M, \\ -2M\,, & \text{if } y \leq -2M. \end{cases} \tag{2.1}$$

Let us set

$$N(u(x)) = \sum_{i=1}^{2} \frac{\partial}{\partial x_i}(H(u_i(x))u(x)) \quad , \quad \forall u \in \mathcal{X}.$$

We observe that

$$\begin{cases} |u|_{\mathcal{L}^\infty(Q)} \leq 2M \\ u \in \mathcal{V} \end{cases} \quad \text{implies} \quad N(u) = B(u).$$

We consider now the following modified scheme in which we replace $B(u)$ in (1.4) by $N(u)$:

$$\begin{cases} \text{Let} \quad u_N^0 = \Pi_N u_0, \quad \text{find} u_N^n \in \mathcal{V}_N \quad \text{such that} \\ \dfrac{1}{k}(u_N^{n+1} - u_N^n, v)_\omega + \dfrac{\nu}{2} a_\omega(u_N^{n+1} + u_N^n, v) \\ \qquad = -(1.5N(u_N^n) - 0.5N(u_N^{n-1}), v)_\omega + < f^{n+\frac{1}{2}}, v >_\omega \quad , \quad \forall v \in \tilde{\mathcal{V}}_N. \end{cases} \tag{2.2}$$

To start (2.1), one needs to know also u_N^1. We assume that u_N^1 is given such that the truncation error at time $t = k$ is $O(t^2)$. Such a u_N^1 may be obtained by for instance the Runge-Kutta scheme.

At each time step, the scheme (2.2) can be interpreted as a discrete Stokes type equation which admits a unique solution as proved in [3] by means of the following lemma which is essential for our analysis.

Lemma 1. *$\forall u \in \mathcal{V}_N$, $\exists\ \phi \in H^2_{0,\omega} \cap S_N$ such that*

$$u = \mathrm{rot}\phi \quad \text{and} \quad \omega^{-1}\mathrm{curl}(\phi\omega) \in \tilde{\mathcal{V}}_N.$$

We define the operator T: $\mathcal{V}_N \to \tilde{\mathcal{V}}_N$ by

$$Tu = \omega^{-1}\mathrm{curl}\ (\phi\omega). \tag{2.3}$$

Then

$$(u, Tu)_\omega = (\ \mathrm{curl}\ \phi,\ \mathrm{curl}\ (\phi\omega)) = a_\omega(\phi,\phi) = ||\phi||^2_\omega \geq \alpha|u|^2_\omega, \tag{2.4}$$

$$a_\omega(u, Tu) = (\Delta\phi, \Delta(\phi\omega)) \geq \beta||\phi||^2_{2,\omega} \geq \beta||u||^2_\omega. \tag{2.5}$$

where curl $\phi = (\frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial x})$.

Let us prove first a stability result.

Lemma 2. *Let $u_0 \in \mathcal{V}_\omega$ and $f \in \mathcal{L}^2(0,T,\mathcal{X}')$. We assume moreover that the solution u of (1.1) satisfies (H1). Then the scheme (2.2) is unconditionally stable. More precisely, for $0 \leq m \leq K$, we have*

$$|u^m_N|^2_\omega + k\nu\sum_{n=0}^{m-1}||u^{n+1}_N + u^n_N||^2_\omega \leq L_1(f, u^1_N, u_0), \tag{2.6}$$

$$\sum_{n=0}^{m-1}|u^{n+1}_N - u^n_N|^2_\omega \leq L_1(f, u^1_N, u_0). \tag{2.7}$$

Proof. In order to alleviate notations, we will ignore the sub-index N whenever no confusion is possible.

Replacing v by $2k \cdot T(u^{n+1} + u^n)$ in (2.2) and using Lemma 1, we obtain

$$\begin{aligned} &2(||\phi^{n+1}||^2_\omega - ||\phi^n||^2_\omega) + k\beta\nu||\phi^{n+1} + \phi^n||^2_{2,\omega} \\ &\qquad = k < 2f^{n+\frac{1}{2}} - 3N(u^n) + N(u^{n-1}), T(u^{n+1} + u^n) >_\omega . \end{aligned} \tag{2.8}$$

where ϕ^n is the function associated to u^n_N (cf. Lemma 1).

We derive from the Schwarz inequality that

$$< 2f^{n+\frac{1}{2}}, v >_\omega \leq 2||f^{n+\frac{1}{2}}||_{\mathcal{X}'}||v||_\omega \leq \frac{\nu\beta}{8}||v||^2_\omega + \frac{8}{\nu\beta}||f^{n+\frac{1}{2}}||^2_{\mathcal{X}'}. \tag{2.9}$$

The modified nonlinear term can be easily handled thanks to the definition of the function H. Actually, we derive from integration by parts and the Schwarz inequality

$$\begin{aligned} (N(u), v)_\omega &= \sum_{i,j=1}^{2}\int_\Omega \frac{\partial}{x_i}(H(u_i)\cdot u_j)v_j\omega dx = -\sum_{i,j=1}^{2}\int_\Omega H(u_i)u_j\frac{\partial(v_j\omega_i)}{\partial x_i}\omega_j dx \\ &\leq 2M\sum_{i,j=1}^{2}\int_{-1}^{1}|u_j|_{\omega_j}\cdot|\frac{\partial(v_j\omega_i)}{\partial x_i}|_{\omega_j}dx_i. \end{aligned} \tag{2.10}$$

Then, by using the following result (cf. [5]):

$$\left|\omega_i^{-1}\frac{\partial(v_j\omega_i)}{\partial x_i}\right|_{\omega_i} \le c\|v_j\|_{\omega_i}$$

we arrive at

$$(N(u),v)_\omega \le c_1 M|u|_\omega\|v\|_\omega. \tag{2.11}$$

We recall that (cf. [9])

$$\int_\Omega \phi^2\omega^5 dx \le \alpha_1\|\phi\|_\omega^2 \quad \text{and} \quad \int_\Omega \phi^2\omega^9 dx \le \beta_1\|\phi\|_{2,\omega}^2 \ , \ \forall\, \phi\in \mathcal{X}. \tag{2.12}$$

One can then readily check that

$$\|T(u^{n+1}+u^n)\|_\omega \le c_2\|\phi^{n+1}+\phi^n\|_{2,\omega}.$$

We infer from (2.9), (2.10) and (2.12) that

$$\begin{aligned}
&|k < 2\ f^{n+\frac12} - 3N(u^n) + N(u^{n-1}), T(u^{n+1}+u^n) >_\omega | \\
&\le \frac{k\nu\beta}{2}\|\phi^{n+1}+\phi^n\|_{2,\omega}^2 + \frac{8k}{\nu\beta}\|f^{n+\frac12}\|_{\mathcal{X}'}^2 \\
&+\frac{c_2M^2k}{\nu}(\|\phi^{n-1}\|_\omega^2 + \|\phi^n\|_\omega^2).
\end{aligned} \tag{2.13}$$

Now summing (2.13) for $n = 1,\dots,m-1$ $(m \le K)$, we get

$$\begin{aligned}
&2\|\phi^m\|_\omega^2 + \frac{k\nu\beta}{2}\sum_{n=1}^{m-1}\|\phi^{n+1}+\phi^n\|_{2,\omega}^2 \\
&\le \frac{8k}{\nu\beta}\sum_{n=1}^{m-1}\|f^{n+\frac12}\|_{\mathcal{X}'}^2 + 2\|\phi^1\|_\omega^2 + \frac{2c_2M^2k}{\nu}\sum_{n=0}^{m-1}\|\phi^n\|_\omega^2.
\end{aligned} \tag{2.14}$$

(2.6) can then be established by using the following simple remark:

Let a_m, b_m and c_m be three positive sequences with c_m increasing and such that

$$a_0 + b_0 \le c_0, \quad a_m + b_m \le c_m + \lambda\sum_{n=0}^{m-1} a_n \ , \ \forall\, m \ge 1$$

with $\lambda > 0$, then

$$a_m + b_m \le \exp(\lambda m)c_m \ , \ \forall\, m \ge 1. \tag{2.15}$$

It is proveed readily by induction that

$$a_m + b_m \le (1+\lambda)^m c_m,$$

and (2.15) follows then from $(1+\lambda)^m \le \exp(\lambda m)$.

We apply (2.15) with

$$a_m = 2\|\phi^m\|_\omega^2 \ , \ b_m = \frac{k\nu\beta}{2}\sum_{n=1}^{m-1}\|\phi^{n+1}+\phi^n\|_{2,\omega}^2$$

and

$$c_m = \frac{8k}{\nu\beta}\sum_{n=1}^{m-1} \|f^{n+\frac{1}{2}}\|^2_{\mathcal{X}'} + 2\|\phi^1\|^2_\omega.$$

By taking into account (2.4) and (2.5), we obtain

$$\begin{aligned} 2\alpha|u^m|^2_\omega &+ \frac{k\nu\beta}{2}\sum_{n=0}^{m-1}\|u^{n+1}+u^n\|^2_\omega \\ &\le \Big(\frac{8}{\nu\beta}\|f\|^2_{\mathcal{L}^2(0,T,\mathcal{X}')} + 2|u^1_N|^2_\omega + \frac{c_2M^2k}{\nu}|u^0_N|^2_\omega\Big)\exp\Big(\frac{c_2M^2T}{\nu}\Big)\ ,\ \forall\, 1\le m\le K. \end{aligned}$$

Similarly, by the aid of the last inequality, we can prove (2.7) if we replace v by $2kT(u_N^{n+1}-u_N^n)$ in (2.2) and proceed exactly as we did for proving (2.6).

In virtue of the previous lemma, we can prove, by using a compactness argument exactly as in §3.5 of [12], the following convergence theorem.

Theorem 1. *Let $u_{k,N}(t)$ be the function from $[0,T]$ in $\mathcal{V}_N$ defined by*

$$u_{k,N}(t) = u_N^m\ ,\quad \forall t\in[mk,(m+1)k),\quad m=0,1,\ldots,K-1.$$

Given $u_0\in\mathcal{V}_\omega$, $f\in\mathcal{L}^2(0,T,\mathcal{X}')$ and we assume that the solution u satisfies (H1). *Then*

$$u_{k,N}(\cdot)\to u(\cdot),\quad when\quad k, N^{-1}\to 0$$

in $\mathcal{L}^2(0,T,\mathcal{V}_\omega)$ weakly, $L^\infty(0,T,\mathcal{H}_\omega)$ weak-star and $\mathcal{L}^2(0,T,\mathcal{H}_\omega)$ strongly.

The proof of this theorem is quite long and technical. We refer to [12] for a detailed presentation.

Remark 1. Theorem 1 also implies that

$$u\in\mathcal{L}^2(0,T,\mathcal{V}_\omega)\cap L^\infty(0,T,\mathcal{H}_\omega) \tag{2.16}$$

where u is the solution of (1.1). Actually, the hypothesis (H1) can be removed if we work directly on the scheme (1.4); we get instead a conditional stability and convergence result, but (2.16) still holds.

§3. Error Estimates

In this section, we will derive first an error estimate by assuming that the solution u is sufficiently smooth. Namely, we assume

(H2) $$u''(t)\in\mathcal{L}^2(0,T,\mathcal{V}_\omega)\ \text{ and }\ u'''(t)\in\mathcal{L}^2(0,T,\mathcal{X}').$$

We then prove that, by using this error estimate and an inequality which controls the L^∞ norm by L^2_ω norm in the discrete space S_N, the schemes (1.4) and (2.2) are equivalent under a very weak condition which we will describe late on.

We establish first a preliminary result explaining the truncation error of the scheme (2.2).

Lemma 3. *We assume that the solution u of* (1.1) *satisfies* (H1) *and* (H2). *Then*

$$k\sum_{n=1}^{K-1}||\varepsilon_n^k||_{X'}^2 \leq L_2(u,f)k^4 \tag{3.1}$$

where ε_n^k is defined by

$$\begin{aligned}<\varepsilon_n^k, v>_\omega = \tfrac{1}{k}(u(n+1)-u(n),v)_\omega + \tfrac{\nu}{2}a_\omega(u(n+1)+u(n),v)\\ -(f^{n+\frac{1}{2}} - 1.5N(u(n)) + 0.5N(u(n-1)),v)_\omega\ ,\ \forall\, v\in \tilde{V}_\omega\end{aligned} \tag{3.2}$$

where we have set $u(n) = u(n\cdot k)$, $n = 0,1,\ldots,K$.

Proof. We develop $u(n-1), u(n)$ and $u(n+1)$ at $t = k(n+\frac{1}{2})$ by using Taylor's formula with integral residue

$$u(n-1) = u(n+\frac{1}{2}) - 1.5ku'(n+\frac{1}{2}) + \tfrac{9}{4}k\int_{k(n-1)}^{k(n+\frac{1}{2})}(t-k(n-1))u''(t)dt,$$

$$u(n) = u(n+\frac{1}{2}) - 0.5ku'(n+\frac{1}{2}) + \tfrac{1}{4}k\int_{kn}^{k(n+\frac{1}{2})}(t-kn)u''(t)dt,$$

$$\begin{aligned}< u(n),v>_\omega &=< u(n+\frac{1}{2}) - 0.5ku'(n+\frac{1}{2}) + \frac{1}{4}k^2u''(n+\frac{1}{2}),v>_\omega\\ &\quad -\frac{1}{8}k^2 < \int_{kn}^{k(n+\frac{1}{2})}(t-k(n+\frac{1}{2}))u'''(t)dt, v>_\omega,\\ < u(n+1),v>_\omega &=< u(n+\frac{1}{2}) + 0.5ku'(n+\frac{1}{2}) + \frac{1}{4}k^2u''(n+\frac{1}{2}),v>_\omega\\ &\quad +\frac{1}{8}k^2 < \int_{k(n+\frac{1}{2})}^{k(n+1)}(t-k(n+\frac{1}{2}))u'''(t)dt, v>_\omega .\end{aligned}$$

A direct computation leads to

$$\begin{aligned}&|(1.5u(n)\cdot\nabla u(n) - 0.5u(n-1)\cdot\nabla u(n-1) - u(n+\frac{1}{2})\cdot\nabla u(n+\frac{1}{2}),v)_\omega|\\ &\quad\leq c_1k\int_{k(n-1)}^{k(n+\frac{1}{2})}||u''(t)||_\omega dt\cdot||v||_\omega,\end{aligned}$$

$$\left|(\frac{1}{k}(u(n+1)-u(n)) - u'(n+\frac{1}{2}),v)_\omega\right| \leq c_2k\int_{kn}^{k(n+1)}||u''(t)||_{X'}dt\cdot||v||_\omega,$$

$$\left|a_\omega(0.5(u(n+1)+u(n)) - u(n+\frac{1}{2}),v)_\omega\right| \leq c_3k\int_{kn}^{k(n+1)}||u'''(t)||_{X'}dt\cdot||v||_\omega.$$

We then substract (2.2) from (3.2). Using the above inequalities and the relation $N(u(n)) =$

$B(u(n))$, we can get

$$\|\varepsilon_n^k\|_{\mathcal{X}'} \le c_4 k \int_{k(n-1)}^{k(n+1)} (\|u'''(t)\|_{\mathcal{X}'} + \|u''(t)\|_\omega) dt.$$

It follows that

$$\|\varepsilon_n^k\|_{\mathcal{X}'}^2 \le c_5 k^3 \int_{k(n-1)}^{k(n+1)} (\|u'''(t)\|_{\mathcal{X}'}^2 + \|u''(t)\|_\omega^2) dt.$$

The summation of the last inequality for $n = 1, \cdots, K-1$ leads to

$$k \sum_{n=1}^{K-1} \|\varepsilon_n^k\|_{\mathcal{X}'}^2 \le c_6 k^4 (\|u'''(t)\|_{\mathcal{L}^2(0,T,\mathcal{X}')}^2 + \|u''(t)\|_{\mathcal{L}^2(0,T,\mathcal{V}_\omega)}^2).$$

We can now prove the following theorem.

Theorem 2. *Assuming that the solution u satisfies* (H1), (H2) *and*

$$u \in C([0,T], \mathcal{H}_\omega^s(\Omega)) \quad , u' \in C([0,T], \mathcal{H}_\omega^{s-1}(\Omega)), \quad s \le 2. \tag{3.3}$$

Then

$$\begin{aligned} |u(m) - u_N^m|_\omega^2 + k\nu \sum_{n=0}^{m-1} \|u(n+1) + u(n) - u_N^n - u_N^{n+1}\|_\omega^2 \\ \le L_3(k^4 + |u_N^1 - \Pi_N u(1)|_\omega^2 + N^{2(1-s)}) \quad , \quad 1 \le m \le K. \end{aligned} \tag{3.4}$$

Proof. Let us set

$$e^n = \Pi_N u(n) - u_N^n \quad \textit{and} \quad \bar{e}_n = u(n) - \Pi_N u(n) \quad \text{for} \quad n = 0, 1, \cdots, K.$$

We can then reformulate (3.2) in the discrete form by the help of $\bar{e}_n$ and Π_N:

$$\begin{aligned} \frac{1}{k}(\Pi_N \ (u(n+1) - u(n)), v)_\omega + \tfrac{\nu}{2} a_\omega(\Pi_N(u(n+1) + u(n)), v) \\ -(f^{n+\frac{1}{2}} - 1.5N(u(n)) + 0.5N(u(n-1)), v)_\omega \\ = < \varepsilon_n^k, v >_\omega - \tfrac{1}{k}(\bar{e}_{n+1} - \bar{e}_n, v)_\omega \ , \ \forall\, v \in \tilde{\mathcal{V}}_N. \end{aligned} \tag{3.5}$$

Using Taylor's formula as we did in the proof of Lemma 3, we can get

$$\begin{aligned} |(\bar{e}_{n+1} - \bar{e}_n, v)_\omega| \le \{|u'(n+\frac{1}{2}) - \Pi_N u'(n+\frac{1}{2})|_\omega + c_1 k \int_{k(n-1)}^{k(n+1)} \|u'''(t)\|_{\mathcal{X}'} dt\} \|v\|_\omega \\ \le |u'(n+\frac{1}{2}) - \Pi_N u'(n+\frac{1}{2})|_\omega^2 + 4c_1 k^3 \int_{k(n-1)}^{k(n+1)} \|u'''(t)\|_{\mathcal{X}'} dt + \frac{1}{2}\|v\|_\omega^2. \end{aligned} \tag{3.6}$$

We subtract (3.5) from (2.2)

$$\begin{aligned} \frac{1}{k}\ (e^{n+1} - e^n, v)_\omega + \frac{\nu}{2} a_\omega(e^{n+1} + e^n, v) = < \varepsilon_n^k, v >_\omega - \frac{1}{k}(\bar{e}_{n+1} - \bar{e}_n, v)_\omega \\ +1.5(N(u(n)) - N(u^n), v)_\omega - 0.5(N(u(n-1)) - N(u^{n-1}), v)_\omega \ , \ \forall\, v \in \tilde{\mathcal{V}}_N. \end{aligned} \tag{3.7}$$

Let us majorise first the nonlinear terms. By definition

$$N(u\ (n)) - N(u^n) = \sum_{i=1}^{2} \frac{\partial}{\partial x_i}(H(u_i(n))u(n) - H(u_i^n)u^n)$$
$$= \sum_{i=1}^{2} \frac{\partial}{\partial x_i}\{u(n)(H(u_i(n)) - H(u_i^n)) + H(u_i^n)(u(n) - u^n)\}.$$

One can readily check from the definition of H that

$$|H(x) - H(y)| \le |x - y| \ ,\ \forall\ x, y \in R.$$

which implies that

$$|H(u) - H(v)|_\omega \le |u - v|_\omega \ ,\ \forall\ u, v \in \mathcal{L}^2_\omega(\Omega). \tag{3.8}$$

We then take the scalar product of (3.7) with v and integrate by parts. Using (3.8), (1.5) and (H1), we obtain:

$$|N(u(n)) - N(u^n), v)_\omega| \le \{2M|u(n) - u^n|_\omega + |u(n)|_{\mathcal{L}^\infty(\Omega)}|u(n) - u^n|_\omega\}||v||_\omega$$
$$\le c_2(|e^n|_\omega + |\bar{e}_n|_\omega)||v||_\omega \le c_2(|e^n|^2_\omega + |\bar{e}_n|^2_\omega) + \tfrac{1}{2}c_2||v||^2_\omega. \tag{3.9}$$

Replacing v by $k \cdot T(e^{n+1} + e^n)$ (where T is the operator defined in Lemma 1) in (3.7) and taking into account (3.6), (3.9) and Lemma 1, we find

$$|e^{n+1}|^2_\omega - |e^n|^2_\omega + \frac{k\nu}{4}||e^{n+1} + e^n||^2_\omega \le c_3 k\{|e^n|^2_\omega + |e^{n+1}|^2_\omega + |\bar{e}_n|^2_\omega + |\bar{e}_{n-1}|^2_\omega + ||e^k_n||^2_{\chi'}$$
$$+|u'(n+\frac{1}{2}) - \Pi_N u'(n+\frac{1}{2})|^2_\omega + k^3\int_{k(n-1)}^{k(n+1)} ||u'''(t)||^2_{\chi'}dt\}.$$

We then sum this inequality for $n = 1, \ldots, m-1$. Using (3.1) and (H2), we get

$$|e^m|^2_\omega + \frac{k\nu}{4}\sum_{n=1}^{m-1} ||e^{n+1} + e^n||^2_\omega \le c_4 k^4 + |e^1|^2_\omega + c_5 k\sum_{n=0}^{m-1} |e^n|^2_\omega$$
$$+c_5 k\sum_{n=1}^{m-1}\{|u'(n+\frac{1}{2}) - \Pi_N u'(n+\frac{1}{2})|^2_\omega + |\bar{e}_n|^2_\omega\}.$$

We can now apply (2.15) to the last inequality, which gives

$$|e^m|^2_\omega + k\nu\sum_{n=1}^{m-1} ||e^{n+1} + e^n||^2_\omega \le c_6 k^4 + |e^1|^2_\omega + c_6 k\sum_{n=0}^{m-1} |\bar{e}_n|^2_\omega$$
$$+c_6 k|e^0|^2_\omega + c_6 k\sum_{n=0}^{m-1}\{|u'(n+\frac{1}{2}) - \Pi_N u'(n+\frac{1}{2})|^2_\omega\}.$$

The proof is complete by combining (1.7), (3.3) and the relations

$$|u(n) - u^n|_\omega \le |e^n|_\omega + |\bar{e}_n|_\omega\ ,$$

$$|\bar{e}_n|_\omega \le c\|\bar{e}_n\|_\omega \quad and \quad e^0 = 0$$

into the last inequality.

By using Theorem 2, we can now prove the following interesting result.

Theorem 3. *We suppose that all the conditions of Theorem 2 are satisfied. Then, there exists $N_0 > 0$ such that the schemes (1.4) and (2.2) are equivalent as long as $N > N_0$ and $kN^{\frac{1}{2}} \to 0$.*

Proof. It is sufficient to prove that

$$\max_{0\le n\le K} |u_N^n - u(n)|_\infty \to 0 \ \ (N \to 0) \ \text{ provided } \ kN^{\frac{1}{2}} \to 0.$$

Let us prove first that

$$|q|_\infty \le N|q|_\omega \ , \ \forall\, q \in S_N,$$

which is a kind of inverse inequality.

We recall the following orthogonality formula satisfied by the Chebyshev polynomials:

$$\int_{-1}^{1} T_k(x)T_l(x)\omega(x)dx = \frac{\pi}{2}c_k\delta_{kl}$$

where $c_0 = 2$ and $c_k = 1$ for $k \ge 1$.

We infer from this formula that

$$\forall q = \sum_{kl=0}^{N} q_{kl}T_k(x)T_l(y), \qquad |q|_\omega^2 = \frac{\pi^2}{4}\sum_{k,l=0}^{N} c_kc_lq_{kl}^2.$$

Since $|T_k(x)| \le 1$, $\forall\, x \in I$, we deduce that

$$|q|_\infty^2 = \Big|\sum_{k,l=0}^{N} q_{kl}T_k(x)T_l(y)\Big|_\infty^2 \le \{\sum_{k,l=0}^{N}|q_{kl}|\}^2 \le (N+1)^2\sum_{k,l=0}^{N}(q_{kl})^2 \le \frac{4}{\pi^2}(N+1)^2|q|_\omega{}^2.$$

We know from Theorem 2 that

$$\max_{m=0,1,\cdots,K} |u_N^m - \Pi_N u(m)|_\omega{}^2 \le c(k^4 + |u_N^1 - \Pi_N u(1)|_\omega{}^2 + N^{2(1-s)}).$$

The last two inequalities imply that

$$\begin{aligned}|u_N^m - u(m)|_\infty^2 &\le 2(|u_N^m - \Pi_N u(m)|_\infty^2 + |u(m) - \Pi_N u(m)|_\infty^2)\\ &\le \frac{8(N+1)^2}{\pi^2}|u_N^m - \Pi_N u(m)|_\omega{}^2 + |u(m) - \Pi_N u(m)|_\infty^2\\ &\le cN^2(k^4 + |u_N^1 - \Pi_N u(1)|_\omega{}^2 + N^{2(1-s)}) + |u(m) - \Pi_N u(m)|_\infty^2. \end{aligned} \tag{3.10}$$

The first term on the right-hand side of (3.10) tends to zero provided that

$$kN^{\frac{1}{2}} \to 0 \ . \tag{3.11}$$

In virtue of the following Sobolev inequality (cf. for instance [1]):

$$|u|_\infty^2 \le c\|u\|_2 \cdot |u| \ , \ \forall\, u \in H^2(\Omega) \tag{3.12}$$

and by taking into account (1.7) and (3.3), the second term on the right-hand side of (3.10) also tends to zero. It means that

$$\max_{0\le n\le K} |u_N^n - u(n)|_\infty \to 0 \ , \quad N \to 0$$

under the condition (3.11).

Consequently, there exists $N_0 > 0$ such that

$$\max_{0\le n\le K} |u_N^n - u(n)|_\infty \le M \ , \ \forall\, N \le N_0.$$

Hence

$$\begin{aligned}\max_{m=0,1,\cdots,K} |u_N^m|_\infty &\le \max_{0\le n\le K} |u_N^n - u(n)|_\infty + \max_{0\le n\le K} |u(n)|_\infty \\ &\le M + M = 2M\end{aligned}$$

which implies $N(u_N^m) = B(u_N^m)$.

Remark 2. By giving more smoothness assumtions on the solution u, we can reduce the very restrictive time step constraint (1.9) for the scheme (1.4) to the very weak condition (3.11) which is evidently not a constraint in practice since we should keep the time step k reasonably small to balance the spectral precision of the space discretization.

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