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NONLINEAR INTEGRAL EQUATION OF INVERSE SCATTERING PROBLEMS OF WAVE EQUATION AND ITERATION*

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Abstract

In this paper, we reduce the 1–D, 2–D and 3–D inverse scattering problems of the wave equation into the nonlinear integral equation. The iteration for solving the above integral equations has been considered.

§1. Introduction

There are several works about the inverse scattering problem of the wave equation [1–5]. In [4] and [5], the characteristic iteration for solving 1–D, 2–D and 3–D inverse scattering problems to determine potential have been given. In this paper, by the method of [4] and [5], we reduce the above inverse problem to the nonlinear integral equation. In the 1–D case, the integral equation is of second kind. In 2–D and 3–D cases, the integral equations are Radon's interesting integral geometry problem.

The integral equation in this paper will be useful for the theoretical and numerical analysis and application of the above inversion.

The iteration for solving the above integral equation is considered. Moreover, we perform several simulative numerical calculations in the 1–D and 2–D scattering inversions and get excellent numerical results.

We shall first deal with the nonlinear integral equation in 3–D. Then we will describe the nonlinear integral equation in 1–D and 2–D. A description of the parallel iteration for solving the above nonlinear integral equations is given in Section 4. Our numerical results are presented and discussed in Section 5.

§2. 3–D Inverse Scattering Potential Problem and Its Nonlinear Integral Equation

2.1 Basic equation and its scattering inversion

$$\frac{\partial^2 u}{\partial t^2} - \left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right] + q(x,y,z)u = 0,$$
$$(x,y,z) \in R^2 \times R^+, \quad t > 0, \tag{2.1}$$
$$u(x,y,z,t) = 0, \qquad (x,y,z) \in R^2 \times R^+, \quad t \le 0, \tag{2.2}$$

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$$\frac{\partial u}{\partial z}(x,y,0,t)=\delta(x,y,t),\quad (x,y)\in R^2,\quad t\geq 0. \tag{2.3}$$

To recover $q(x,y,z)$ from measured data on the surface boundary

$$u(x,y,0,t)=f(x,y,t)=-\frac{1}{2\pi}\frac{\delta(t-\sqrt{x^2+y^2})}{\sqrt{x^2+y^2}}+f_S(x,y,t) \tag{2.4}$$

will be called the inverse scattering potential problem of 3–D wave equation, where $\delta(\cdot)$ is the generalized delta function, and $q(x,y,z)\geq 0$ is a continuous function.

2.2 The properties of the solution

Lemma 2.1. *For $q(x,y,z)\in C(R^2\times R^+)$, the solution of (2.1)–(2.3) can be decomposed to*

$$u(x,y,z,t)=-\frac{1}{2\pi}\frac{\delta(t-\sqrt{x^2+y^2+z^2})}{\sqrt{x^2+y^2+z^2}}+v(x,y,z,t), \tag{2.5}$$

where $v(x,y,z,t)$ satisfies

$$\frac{\partial^2 v}{\partial t^2}-\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right)+q(x,y,z)v=\frac{1}{2\pi}q(x,y,z)\frac{\delta(t-\sqrt{x^2+y^2+z^2})}{\sqrt{x^2+y^2+z^2}},$$

$$(x,y,z)\in R^2\times R^+,\quad t>0, \tag{2.6}$$

$$v(x,y,z,t)=0,\quad (x,y,z)\in R^2\times R^+,\quad t\leq 0, \tag{2.7}$$

$$\frac{\partial v}{\partial z}(x,y,0,t)=0,\quad (x,y)\in R^2,\quad t\geq 0. \tag{2.8}$$

Proof. Let

$$u_1=-\frac{1}{2\pi}\frac{\delta(t-\sqrt{x^2+y^2+z^2})}{\sqrt{x^2+y^2+z^2}}.$$

Since u_1 is a solution of (2.1)–(2.3) when $q(x,y,z)=0$, by substituting (2.5) into (2.1)–(2.3), (2.6)–(2.8) can be obtained immediately.

2.3 Nonlinear Integral Equation of 3–D

Theorem 2. *Suppose that $q(x,y,z)\in C(R^2\times R^+)$. Then the 3–D inverse scattering problem for determining $q(x,y,z)$ from (2.1)–(2.4) will be reduced to the following nonlinear integral equation:*

$$\frac{1}{2\pi^2[t^2-(x^2+y^2)]}\iint_{S[(x,y);t]}[q(\xi,\eta,\varsigma)(\xi^2+\eta^2+\varsigma^2)\sin\theta]d\theta d\phi$$
$$=\frac{1}{\pi}\iint_{D[(x,y);t]}q(\xi,\eta,\varsigma)\frac{v(\xi,\eta,\varsigma;t-\sqrt{(\xi-x)^2+(\eta-y)^2+\varsigma^2})}{\sqrt{(\xi-x)^2+(\eta-y)^2+\varsigma^2}}d\xi d\eta d\varsigma+f_S(x,y,t), \tag{2.9}$$

where $S[(x,y);t]$ denotes the half ellipsoid and $D[(x,y);t]$ is its body.

Proof. By [4], [5] convolute both sides of (2.5) about t by

$$w(x,y,z,t;\xi,\eta)=\frac{1}{2\pi}\frac{\delta(t-\sqrt{(x-\xi)^2+(y-\eta)^2+z^2})}{\sqrt{(x-\xi)^2+(y-\eta)^2+z^2}}, \tag{2.10}$$

and then integrate on both sides of (1.5)

$$\begin{aligned}&\int_0^\infty\int_{-\infty}^\infty\int_{-\infty}^\infty\Big[\frac{\partial^2 v}{\partial t^2}-\Big(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\Big)+q(x,y,z)v\Big]*_t w dxdydz\\&=\frac{1}{2\pi}\int_0^\infty\int_{-\infty}^\infty\int_{-\infty}^\infty\Big[q(x,y,z)\frac{\delta(t-\sqrt{x^2+y^2+z^2})}{\sqrt{x^2+y^2+z^2}}*_t w\Big]dxdydz.\end{aligned} \tag{2.11}$$

Integrating by part and by [6], we have

$$\begin{aligned}&\int_{-\infty}^\infty\int_{-\infty}^\infty v*_t\frac{\partial w}{\partial z}dxdy+\int_0^\infty\int_{-\infty}^\infty\int_{-\infty}^\infty q(x,y,z)v*_t wdxdydz\\&=-\frac{1}{4\pi^2}\int_0^\infty\int_{-\infty}^\infty\int_{-\infty}^\infty q(x,y,z)\{\delta(t-\sqrt{x^2+y^2+z^2}-\sqrt{(x-\xi)^2+(y-\eta)^2+z^2})\\&\times(\sqrt{x^2+y^2+z^2}\sqrt{(x-\xi)^2+(\eta-y)^2+z^2})^{-1}\}dxdydz.\end{aligned} \tag{2.12}$$

Changing the positions of (x,y,z) and (ξ,η,ς) in (2.11) and using the property of w, we have

$$\begin{aligned}&-v(x,y,0,t)-\frac{1}{2\pi}\int_0^\infty\int_{-\infty}^\infty\int_{-\infty}^\infty\frac{q(\xi,\eta,\varsigma)v(\xi,\eta,\varsigma,t-\sqrt{(\xi-x)^2+(\eta-y)^2+\varsigma^2})}{\sqrt{(\xi-x)^2+(\eta-y)^2+\varsigma^2}}d\xi d\eta d\varsigma\\&=-\frac{1}{4\pi^2}\int_0^\infty\int_{-\infty}^\infty\int_{-\infty}^\infty q(\xi,\eta,\varsigma)\frac{\delta(t-\sqrt{\xi^2+\eta^2+\varsigma^2}-\sqrt{(\xi-x)^2+(\eta-y)^2+\varsigma^2})}{\sqrt{\xi^2+\eta^2+\varsigma^2}\sqrt{(\xi-x)^2+(\eta-y)^2+\varsigma^2}}d\xi d\eta d\varsigma.\end{aligned} \tag{2.13}$$

Introduce spherical coordinates

$$r=\sqrt{\xi^2+\eta^2+\varsigma^2},\quad r_1=\sqrt{(\xi-x)^2+(\eta-y)^2+\varsigma^2}. \tag{2.14}$$

Then

$$r+r_1=t\quad\text{and}\quad\varsigma>0. \tag{2.15}$$

Denote the half ellipsoid of revolution by the ellipse. The above half ellipsoid and its body will be denoted by $S[(x,y);t]$, and $D[(x,y);t]$ respectively.

Finally, we have

$$\begin{aligned}&\frac{1}{2\pi^2}\frac{1}{t^2-(x^2+y^2)}\iint_{S[(x,y);t]}[q(\xi,\eta,\varsigma)(\xi^2+\eta^2+\varsigma^2)\sin\theta]d\theta d\phi\\&=\frac{1}{\pi}\iiint_{D[(x,y);t]}q(\xi,\eta,\varsigma)\frac{v(\xi,\eta,\varsigma,t-\sqrt{(\xi-x)^2+(\eta-y)^2+\varsigma^2})}{\sqrt{(\xi-x)^2+(\eta-y)^2+\varsigma^2}}d\xi d\eta d\varsigma+f_S(x,y,t).\end{aligned}$$

§3. Nonlinear Integral Equation of Inverse Scattering Potential Problems of 1-D and 2-D

3.1 The inverse scattering problems in 1-D and 2-D

Just as in the 3-D case, the inverse scattering potential problems in 1-D and 2-D are to recover $q(x)$ in 1-D and $q(x,y)$ in 2-D from measured data on the boundary. Note that the point impulse source on the boundary in 1-D is

$$\frac{\partial u}{\partial x}(0,t)=\delta(t),\quad t\ge 0, \tag{3.1}$$

and in 2-D is

$$\frac{\partial u}{\partial y}(x,0,t)=\delta(x,t),\quad (x,t)\in R\times R^+. \tag{3.2}$$

The measured data on the boundary in 1-D is

$$u(0,t)=-H(t)+f_S(t),\quad t\ge 0, \tag{3.3}$$

and in 2-D is

$$u(x,0,t)=-\frac{1}{\pi}\frac{H(t-|x|)}{\sqrt{t^2-|x^2|}}+f_S(x,t),\quad (x,t)\in R\times R^+. \tag{3.4}$$

3.2 Nonlinear Integral Equation of Inverse Scattering Problems in 1-D and 2-D

Theorem 2. *Suppose that $q(x)\in C(0,\infty)$, then the inverse scattering problem for determining $q(x)$ in 1-D can be reduced to the following nonlinear integral equation*

$$q(x)-2\int_0^x q(\xi)w(\xi,2x-\xi)d\xi=2\frac{\partial^2 f_s}{\partial x^2}(2x), \tag{3.5}$$

where $w(x,t)$ satisfies

$$\frac{\partial^2 w}{\partial t}-\frac{\partial^2 w}{\partial x^2}+q(x)w=q(x)\delta(t-x),\quad t\ge 0,\quad x>0, \tag{3.6}$$

$$w(x,t)=0,\quad t\le 0,\quad x\ge 0, \tag{3.7}$$

$$\frac{\partial w}{\partial x}(0,t)=0,\quad t\ge 0. \tag{3.8}$$

Proof. Omitted.

Theorem 3. *Suppose that $q(x,y)\in C(R\times R^+)$, then the 2-D inverse scattering potential problem will be reduced to*

$$\frac{1}{\pi^2}\iint_{D[x,t]}q(\xi,\eta)\int_{\sqrt{(\xi-x)^2+\eta^2}}^{t-\sqrt{\xi^2+\eta^2}}\frac{d\tau}{\sqrt{(t-\tau)^2-(\xi^2+\eta^2)}\sqrt{\tau^2-(\xi-x)^2+\eta^2}}d\xi d\eta$$
$$=\frac{1}{\pi}\iint_{D[x,t]}q(\xi,\eta)\int_{\sqrt{(\xi-x)^2+\eta^2}}^{t-\sqrt{\xi^2+\eta^2}}\frac{v(\xi,\eta,t-\tau)}{\sqrt{\tau^2-[(\xi-x)^2+\eta^2]}}d\tau d\xi d\eta+f_s(x,t), \tag{3.9}$$

where $D[x,t]$ denotes the half disk of the ellipse, and $v(x,y,t)$ satisfies

$$\frac{\partial^2 v}{\partial t^2} - \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) + q(x,y)v = \frac{1}{\pi} q(x,y) \frac{H(t-\sqrt{x^2+y^2})}{\sqrt{t^2-(x^2+y^2)}},$$
$$(x,y) \in R \times R^+,\ t > 0, \tag{3.10}$$
$$v(x,y,t) = 0, \qquad (x,y) \in R \times R^+,\ t \leq 0, \tag{3.11}$$
$$\frac{\partial v}{\partial y}(x,0,t) = 0, \qquad x \in R,\ t \geq 0. \tag{3.12}$$

§4. Iteration

We will describe the iteration for solving the 3–D inverse scattering potential problem of the wave equation. The iterations of 1–D and 2–D cases are similar.

4.1 Iteration for solving 3–D scattering inversion

(i) Pick up $q_0(x,y,z)$, for example, $q_0(x,y,z) = 0$.

(ii) By induction, suppose that $q_n(x,y,z)$ has been obtained. Upon substituting it into (4.5)–(4.7), $v_n(x,y,z,t)$ can be calculated.

(iii) Let G denote the integral geomtry operator

$$G_q = \frac{1}{2\pi^2[t^2-(x^2+y^2)]} \iint_{S[(x,y);t]} [q(\xi,\eta,\varsigma)(\xi^2+\eta^2+\varsigma^2)\sin\theta]\, d\theta d\phi. \tag{4.1}$$

Solve the following linear regularizing[6] integral geometry equation

$$[G+\alpha I]\delta q = v_{(n)}(x,y,0,t) - f_S(x,y,t). \tag{4.2}$$

Here δ_q is the increament function of $q(x,y,z), f_S(x,y,t)$ is the boundary response which is defined in (2.4). α is a regularizing factor which is chose by qnasi-optimality technology.

(iv) Update $q_{(n)}$ to $q_{(n+1)}$:

$$q_{(n+1)} = q_{(n)} + \delta q(x,y,t). \tag{4.3}$$

The process (i)–(iv) is the general iteration. Acoording to that the integral geometry operator G defined by (4.1) preverses symmetry, then the integral geometry equation (4.2) is decomposed into four sub-operator equations in subdomains. By an advanced extrapolate parallel technology we decompose the scattering wave equation (4.5)–(4.7) into four sub-equations using an implict scheme. The above iterative process (i)–(iv) is decomposed into four processes and the parallel iterative process is constructed.

4.2 Iteration for solving 2–D scattering inversion

The iterative program is similar to the 3–D, but Radon's integral equation in (iii) should be replaced by (3.9) which will be reduced to a pair of 1–D Volterra integral equation

$$\int_0^t F_n(x,\tau)d\tau = \frac{1}{\pi}\iint_{D[x,t]} q_n(\xi,\eta) \int_{\sqrt{(\xi-x)^2+\eta^2}}^{t-\sqrt{\xi^2+\eta^2}} \frac{v_n(\xi,\eta,t-\tau)}{\sqrt{\tau^2-[(\xi-x)^2+\eta^2]}} d\tau d\xi d\eta + f_s(x,t), \tag{4.2}$$

and 2-D generalized Radon's integral equation

$$\frac{\sqrt{t^2-x^2}}{8\pi}\int_0^\pi q_{n-1}\Big(\frac{t^2-|x|^2}{2(t+|x|\cos\theta)}\cos\theta, \frac{t^2-|x|^2}{2(t+|x|\cos\theta)}\sin\theta\Big)\frac{\sqrt{t^2+2tx\cos\theta+x^2}}{(t+x\cos\theta)^2}d\theta$$
$$= -\frac{t}{\pi^2}\iint_{D[x,t]} q_n(\xi,\eta)\frac{1}{[t^2-(\sqrt{\xi^2+\eta^2}-\sqrt{(\xi-x)^2+\eta^2})^2]^{\frac{3}{2}}}$$
$$\times \mathbf{K}\Big(\frac{[t^2-(\sqrt{\xi^2+\eta^2}+\sqrt{(\xi-x)^2+\eta^2})^2]^{\frac{1}{2}}}{[t^2-(\sqrt{\xi^2+\eta^2}-\sqrt{(\xi-x)^2+\eta^2})^2]^{\frac{1}{2}}}\Big)d\xi d\eta$$
$$+\frac{t}{\pi^2}\iint_{D[x,t]} q_n(\xi,\eta)\mathbf{K}'\Big(\frac{[t^2-(\sqrt{\xi^2+\eta^2}+\sqrt{(\xi-x)^2+\eta^2})^2]^{\frac{1}{2}}}{[t^2-(\sqrt{\xi^2+\eta^2}-\sqrt{(\xi-x)^2+\eta^2})^2]^{\frac{1}{2}}}\Big)$$
$$\times\frac{1}{t^2-(\sqrt{\xi^2+\eta^2}-\sqrt{(\xi-x)^2+\eta^2})^2}$$
$$\times\frac{[t^2-(\sqrt{\xi^2+\eta^2}+\sqrt{(\xi-x)^2+\eta^2})^2]^{\frac{1}{2}}-[t^2-(\sqrt{\xi^2+\eta^2}-\sqrt{(\xi-x)^2+\eta^2})^2]^{\frac{1}{2}}}{[t^2-(\sqrt{\xi^2+\eta^2}-\sqrt{(\xi-x)^2+\eta^2})^2]^{\frac{3}{2}}[t^2-(\sqrt{\xi^2+\eta^2}+\sqrt{(\xi-x)^2+\eta^2})^2]^{\frac{1}{2}}}d\xi d\tau$$
$$+F_n(x,t), \qquad (x,t)\in R\times R^+, \tag{4.5}$$

where $\mathbf{K}$ is the entire elliptic integral, and $\mathbf{K}'$ is the derivative of $\mathbf{K}$.

§5. Numerical Results

We have performed several numerical simulations in the 1–D and 2–D scattering inversions and got excellent numerical results.

5.1 Numerical result of 1–D case

Numerical results of some tests of 1–D inversion are presented in Fig 1–3. All of the results are excellent and show that iteration (TCC) of 1–D is globally convergent [8]. In this test, the boundary impulse response is presented in Fig 1; in Fig 2, the solid line denotes the exact solution, the solid line with star denotes the approximate solution of the first iteration, and the broken line denotes the initial guess value. In Fig 3, the solid line denoting the exact solution and the solid line with star denoting the sixth iterative solution are coincident, which means that the sixth iterative solution is coincident with the exact solution.

5.2. Numerical result of 2–D case

The numerical simulative test of the iteration (TCC) for solving the 2–D scattering inverse problem has been done. We reduced integral equation (3.9) to the 1–D Volterra equation (4.2) and 2–D generalized Radon's integral geometry equation (4.3) in Lobachevski's hyperbolic geometry space. We solved (4.2) and (4.3) numerically and by using Tikholov's regularizing method [6]. In Fig. 6, the exact solution of $q(x,y)$ is denoted by the solid line, and the approximation of the ninth iteration is denoted by the broken line. The boundary impulse scattering response is presented in Fig. 4, and the first iterative result is presented in Fig. 5.

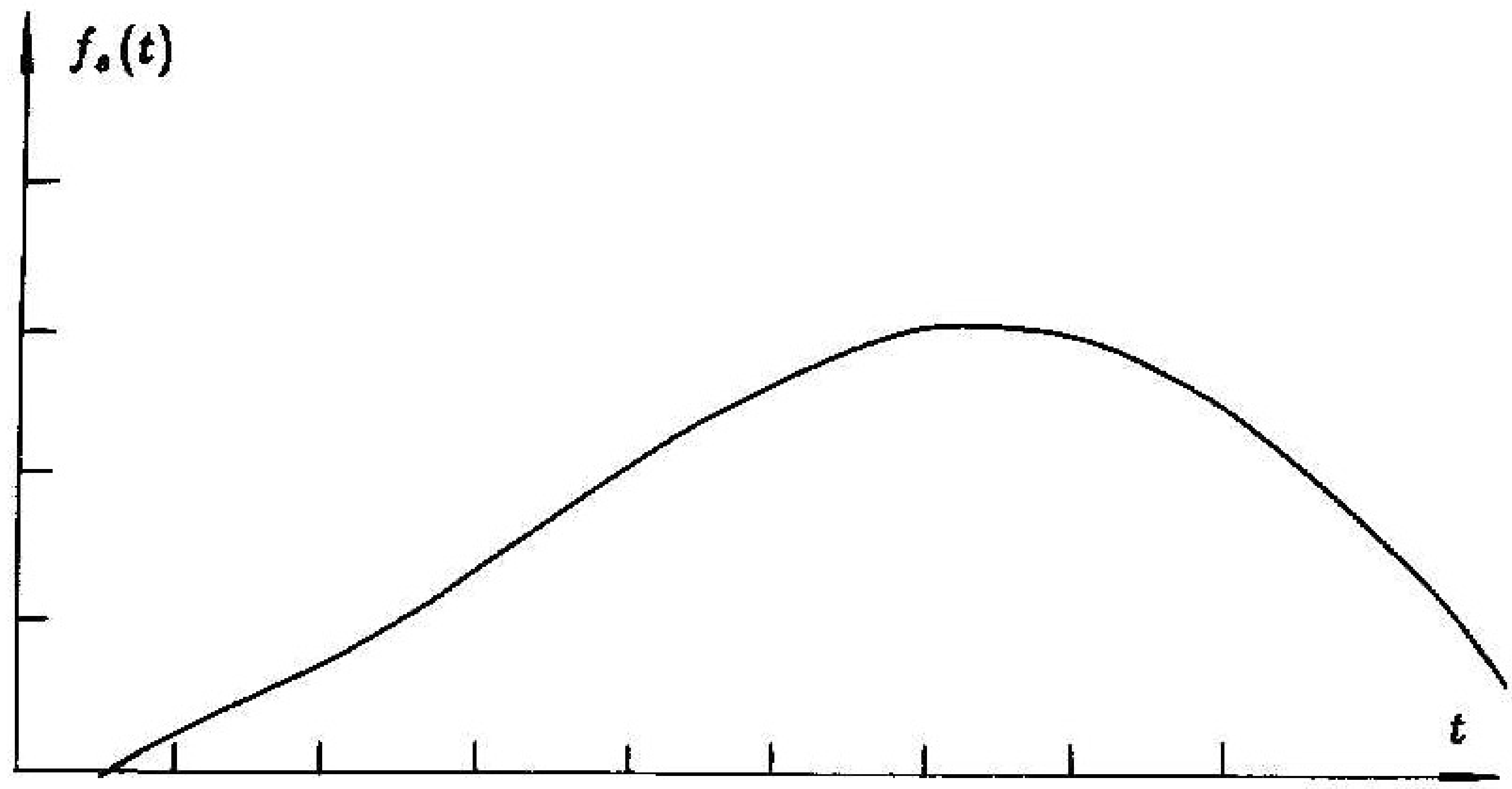

Fig. 1. Boundary scattering response of 1–D case

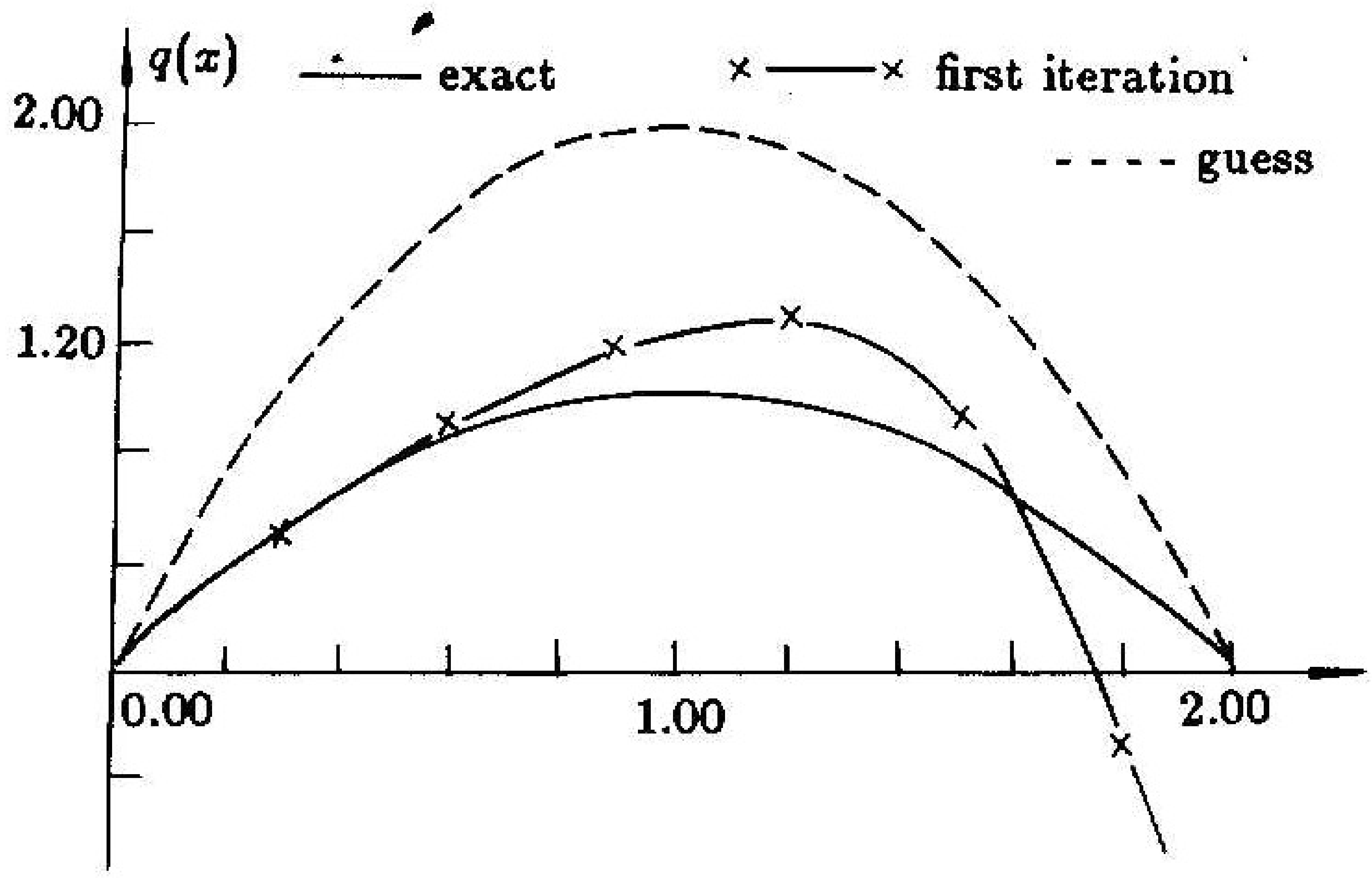

Fig. 2. Numerical results of 1–D case

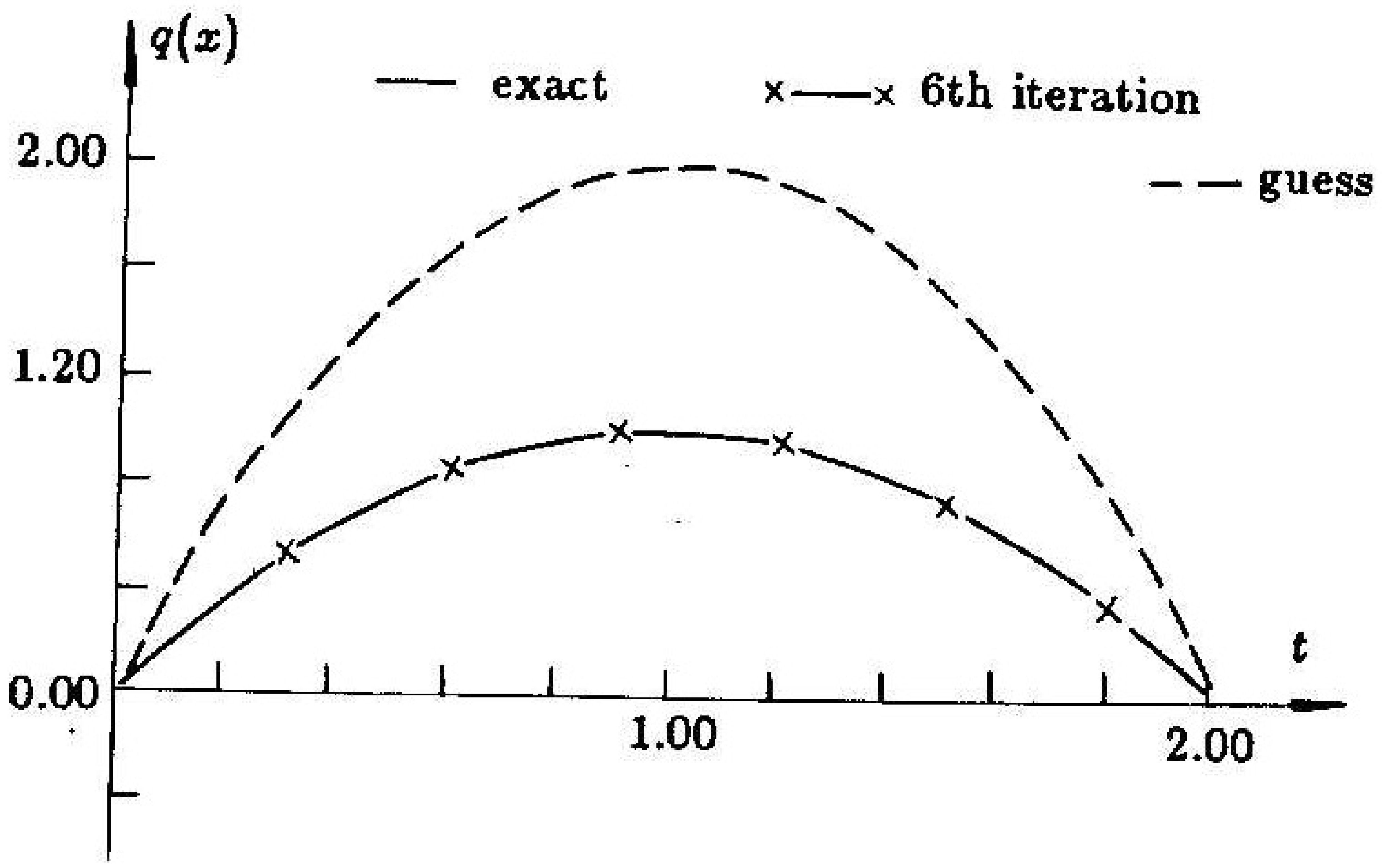

Fig. 3. Numerical results of 1–D case

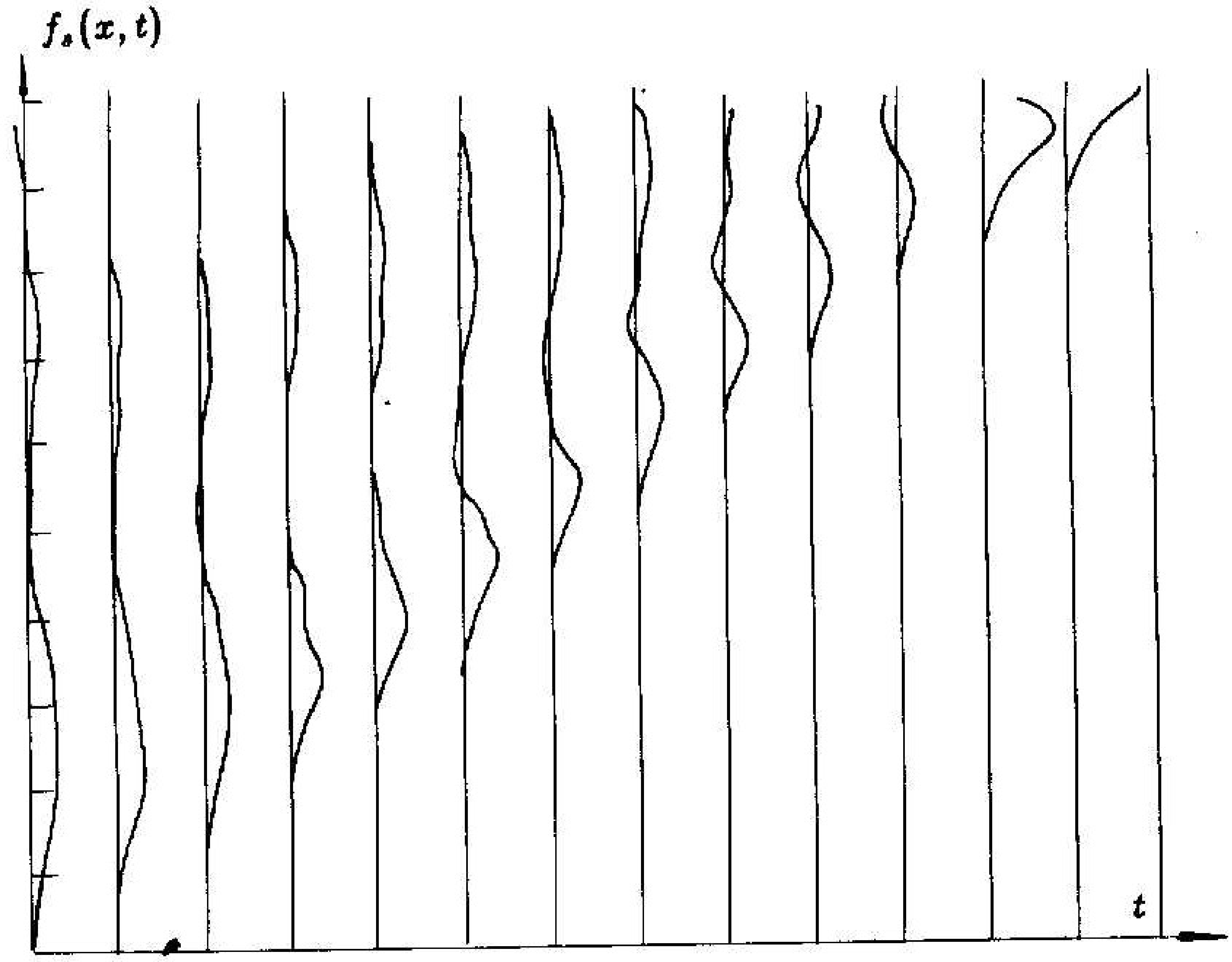

Fig. 4. Boundary impulse scattering response of 2-D case

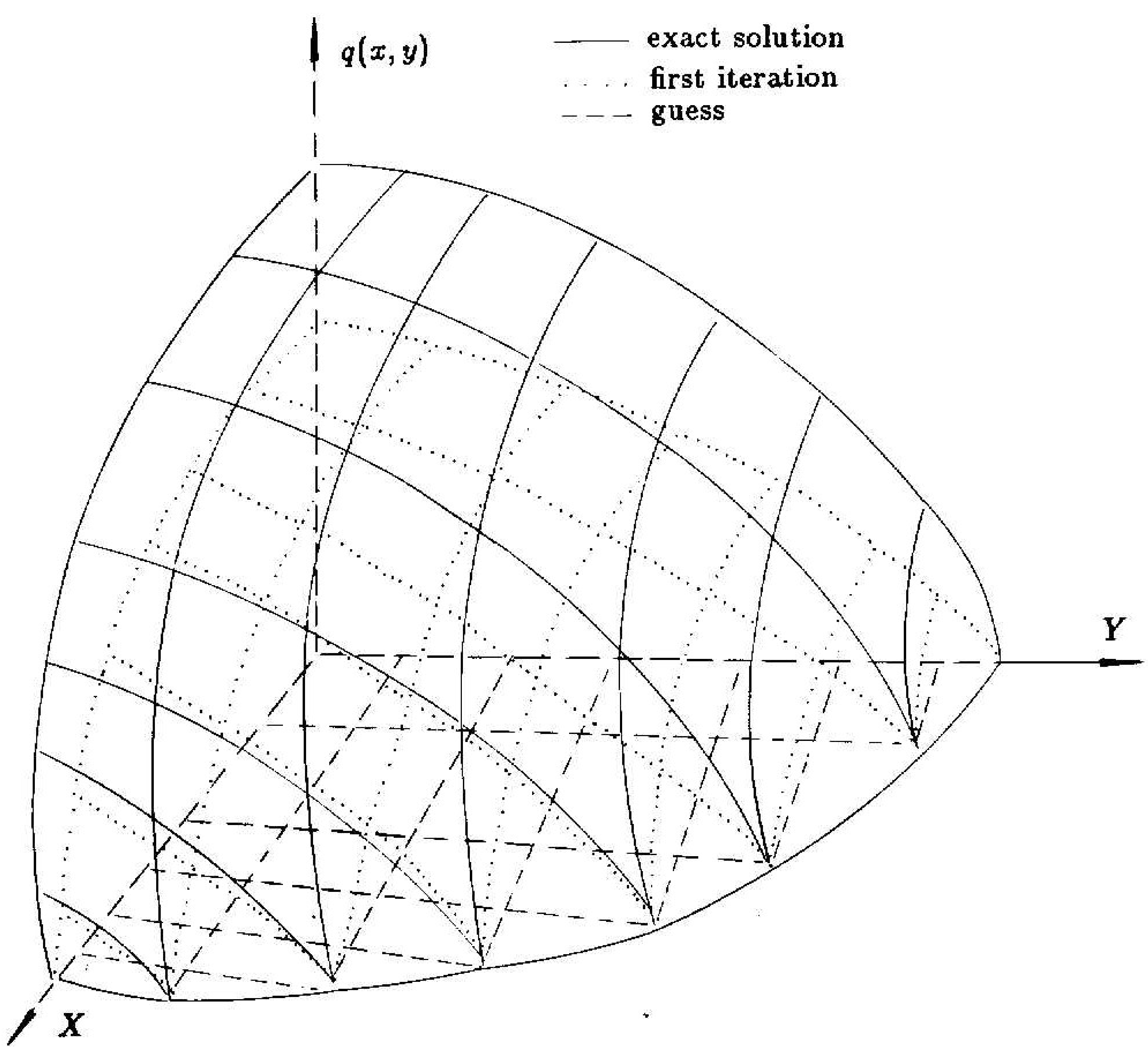

Fig. 5. Numerical result of 2-D case

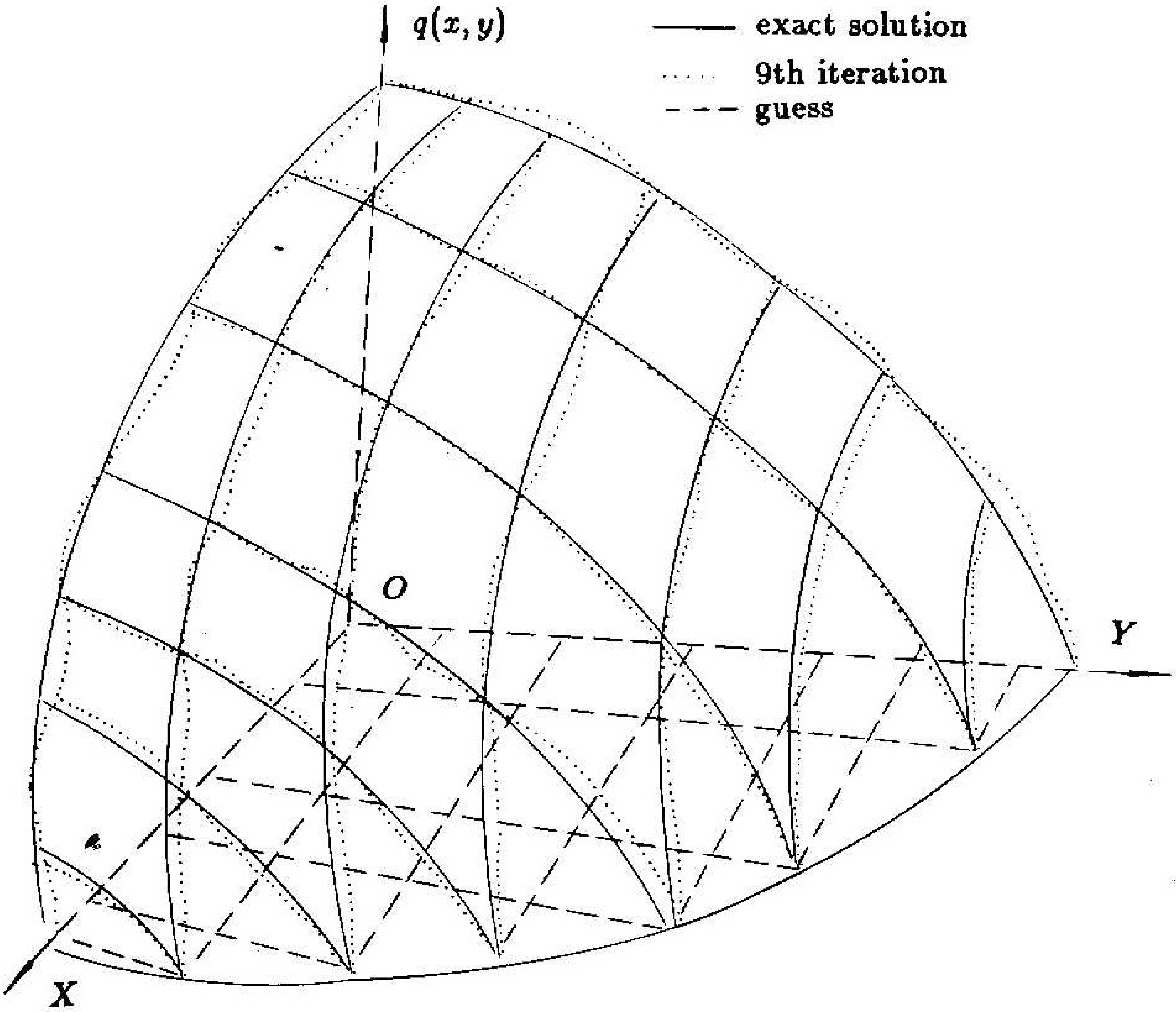

Fig. 6. Numerical result of 2–D case

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References

[1] A. Bamberger, G. Chavent and P. Lailly, About the stability of the inverse problem in 1–D wave equation-application to the interpretation of seismic profiles, *Applied Math. Option,* **5** (1979), 1–47.

[2] K.P.Bube and R.Burridge, The one-dimensional inverse problem of reflection seismology, *SIAM Review,* **25** : 4 (1983), 497–559.

[3] C.S.Morawetz and G.A.Kriegsman, The calculations of an inverse potential problem, *SIAM J. Appl. Math.,* **43** : 4 (1983), 844–854.

[4] Xie Gan–quan, A new iterative method for solving the coefficient inverse problem of the wave equation, *Communication on Pure and Applied Mathematics,* **39** (1986), 307–322.

[5] Xie Gan–quan and Li Jian–hua, A new characteristic iterative method for solving scattering potential inverse problem of 3–D wave equation, *Scientia Sinica,* 1, 1988.

[6] A.N. Tikhonov and V.Y. Arsenin, Solution of Ill-Posed Problems, John Wiley & Sons, 1977.

[7] Xie Gan–quan and Li Jian–hua, Nonlinear integral equation of the coefficient inverse problem of the acoustic wave equation and its iteration, *Scientia Sinica,* 3, 1988.

[8] Xie Gan–quan, L.J.Tang, W. Liu and W. Li, Numerical solution for solving The nonlinear integral equation of one dimensional inverse scattering problem. to appear Journal Computational Mathematics.