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FINITE ELEMENT ANALYSIS FOR A CLASS OF SYSTEM OF NONLINEAR AND NON-SELF-ADJOINT SCHRÖDINGER EQUATIONS*

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Abstract

Because the nonlinear Schrödinger equation is met in many physical problems and is applied widely, the research on well-posedness for its solution and numerical methods has aroused more and more interest. The self-adjoint case has been considered by many authors [1-5]. For a class of system of nonlinear and non-self-adjoint Schrödinger equations which refers to excitons occurring in one dimensional molecular crystals and in α spiral biomolecules, Guo Boling studied in [6], the pure initial and periodic initial value problems of this system and obtained the existence and uniqueness of its solution. In [7] we discussed the difference solution of this system and obtained its error estimate. In this paper, we shall study the finite element method for the periodic initial value problem of this system. Just as Guo pointed out in [6], since it has a non-self-adjoint term, it not only brings about trouble in mathematics, but also creates more difficulty in numerical analysis.

Our analysis will show that for this system, in theoretically we can obtain the same results as when it has no non-self-adjoint term.

§1. Notations and Statement of the Problem

Write $I=[0, 2\pi]$. Let $L_2=L_2(I)$ denote the set of complex valued functions which are square integrable. The scalar product of f and g in $L_2(I)$ is denoted by $(f, g)=\int_I f\bar{g}\,dx$ and the norm of f by $\|f\|_{L_2}=\left(\int_I |f|^2dx\right)^{\frac{1}{2}}$; $L_\infty=L_\infty(I)$ denotes the set of essentially bounded complex value functions, the norm is defined as $\|f\|_{L_\infty}=\operatorname{esssup}_{x\in I}|f(x)|$, and $\tilde{H}^r(I)$ denotes the space of complex value periodic functions, which have square integrable generalized derivatives $D^kf(x)$ $(k\leqslant r)$. Let $\|f\|_r=(\sum_{k\leqslant r}\|D^kf\|^2_{L_2})^{\frac{1}{2}}$ denote the norm of f in $\tilde{H}^r(I)$. The space $L^\infty(0, T; \tilde{H}^r)$ consists of complex value functions $u(x, t)$ that, as a function of x, belong to $\tilde{H}^r$ and that satisfy $\sup_{0\leqslant t\leqslant T}\|u(\cdot,t)\|_r<\infty$. In particular, if $r=0$, it is $L^\infty(0, T; \tilde{L}_2)$, where $\tilde{L}_2$ is the set of periodic functions in L_2.

Now we consider the following periodic initial value problem for the system of nonlinear and non-self-adjoint Schrödinger equations

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$$\begin{cases} i\dfrac{\partial \boldsymbol{u}}{\partial t}-\dfrac{\partial}{\partial x}\left(A(x)\dfrac{\partial \boldsymbol{u}}{\partial x}\right)+B\dfrac{\partial \bar{\boldsymbol{u}}}{\partial x}+\beta(x)q(|\boldsymbol{u}|^2)\boldsymbol{u}+K(x,t)\boldsymbol{u} \\ \quad =\boldsymbol{G}(x,t), \quad -\infty<x<+\infty,\ 0<t<T, \\ \boldsymbol{u}(x,0)=\boldsymbol{u}^0(x), \quad -\infty<x<+\infty, \\ \boldsymbol{u}(x+2\pi,t)=\boldsymbol{u}(x,t), \quad 0<t<T, \end{cases} \tag{1.1}$$

where

$$A(x)=\begin{pmatrix} a_1(x) & & & \\ & a_2(x) & & \\ & & \ddots & \\ & & & a_N(x) \end{pmatrix}, \quad B=\begin{pmatrix} \alpha_1 & & & \\ & \alpha_2 & & \\ & & \ddots & \\ & & & \alpha_N \end{pmatrix},$$

$$K(x,t)=\begin{pmatrix} K_{11}(x,t), & K_{12}(x,t), & \cdots, & K_{1N}(x,t) \\ K_{21}(x,t), & K_{22}(x,t), & \cdots, & K_{2N}(x,t) \\ \cdots & \cdots & \cdots & \cdots \\ K_{N1}(x,t), & K_{N2}(x,t), & \cdots, & K_{NN}(x,t) \end{pmatrix},$$

$\boldsymbol{u}=\boldsymbol{u}(x,t)=(u_1(x,t), u_2(x,t),\cdots, u_N(x,t))^T$ is an unknown complex vector value function, $\bar{\boldsymbol{u}}$ is the conjugate of $\boldsymbol{u}$, $|\boldsymbol{u}|^2=\sum_{i=1}^{N}|u_i|^2$, $\boldsymbol{G}(x,t)=(G_1(x,t), G_2(x,t), \cdots, G_N(x,t))^T$, $G_l(x,t)$ are known complex value functions; $a_l(x)$, $\beta(x)$, $q(s)$, $K_{lr}(x,t)$ are known real value function, $\boldsymbol{u}^0(x)=(u_1^0(x), u_2^0(x), \cdots, u_N^0(x))^T$ is a given complex vector value function; α_l are constants. Besides, these functions satisfy the following conditions:

$$\begin{aligned} &0<m\leqslant a_l(x)\leqslant M,\ |a_l(x)|,\ |a_l'(x)|\leqslant K; \\ &K_{lr}=K_{rl};\ |K_{lr}(x,t)|,\ \left|\frac{\partial K_{lr}}{\partial t}\right|\leqslant K; \\ &\sup_{0<t<T}\int_I |G_l(x,t)|^2dx\leqslant G,\ \left|\frac{\partial G_l}{\partial t}\right|\leqslant G;\ l,r=1,2,\cdots,N, \\ &|q(s)|\leqslant As,\ s>0,\ |\beta(x)|\leqslant B_0, \end{aligned} \tag{A}$$

where $m, M, \cdots, G$ are positive constants.

By the results of [6], we know that (1.1) has solution $\boldsymbol{u}(x,t)$. Moreover $\boldsymbol{u}(x,t)\in L^\infty(0,T;\tilde{H}^2)$, $\boldsymbol{u}_t(x,t)\in L^\infty(0,T;\tilde{L}_2)$ and $\boldsymbol{u}(x,t)$ satisfies

$$\begin{cases} i(u_{lt},v)+\left(a_l\dfrac{\partial u_l}{\partial x},\dfrac{\partial v}{\partial x}\right)+\left(\alpha_l\dfrac{\partial \bar{u}_l}{\partial x}, v\right)+(\beta(x)q(|\boldsymbol{u}|^2)u_l,v)+\sum_{j=1}^{N}(K_{lj}u_j,v) \\ \quad =(G_l(\cdot,t),v), \\ (u_l(\cdot,0),v)=(u_l^0,v),\ \forall v\in\tilde{H}^2,\ l=1,2,\cdots,N. \end{cases} \tag{1.2}$$

In the following, we write $A_l(u,v)=\int_I a_l(x)\dfrac{\partial u}{\partial x}\dfrac{\partial \bar{v}}{\partial x}dx$, and $A(\boldsymbol{u},\boldsymbol{v})=\sum_{l=1}^{N}A_l(u_l,v_l)$.

§ 2. The Semi-Discrete Finite Element Method and Its Error Estimation

We adopt the finite element method to find the approximate solution of (1.2).

We take a finite dimensional subspace U_h+iU_h of $\widetilde{H}^2$, which is composed of piecewise polynomials of degree $r-1$. Here U_h is spanned by real basis functions $\{w_j\}_{j=1}^M$. The finite element solution $\boldsymbol{u}_h\in(U_h+iU_h)^N$ of (1.2) satisfies

$$\begin{cases} i(u_{hlt},\ \chi)+A_l(u_{hl},\ \chi)+(\alpha_l\bar{u}_{hlx},\ \chi)+(\beta(x)q(|\boldsymbol{u}_h|^2)u_{hl},\ \chi)+\sum\limits_{j=1}^{N}(K_{lj}u_{hj},\chi) \\ \quad =(G_l(\cdot,\ t),\ \chi), \\ (u_{hl}(\cdot,\ 0),\ \chi)=(u_l^0,\ \chi),\ \forall\chi\in U_h+iU_h,\ l=1,\ 2,\ \cdots,\ N. \end{cases} \tag{2.1}$$

Lemma 1 (*Sobolev estimation*) [8]. *Suppose $u\in H^k(I)$. Then for any $\varepsilon>0$ and integer $l\geqslant0$, there exists a constant C which depends on l and ε such that*

$$\|D^lu\|_{L_2}\leqslant\varepsilon\|D^ku\|_{L_2}+C\|u\|_{L_2},\quad l<k,$$
$$\|D^lu\|_{L^\infty}\leqslant\varepsilon\|D^ku\|_{L_2}+C\|u\|_{L_2},\quad l<k.$$

Lemma 2[9]. *For any $u\in W_m^1(\Omega)$, $m\geqslant1$ and $r\geqslant1$, there exists a constant $\beta(\Omega)$ such that*

$$\|u\|_{L_p(\Omega)}\leqslant\beta(\Omega)\|u\|^{\alpha}_{W_m^1(\Omega)}\|u\|^{1-\alpha}_{L_r(\Omega)},$$

where $\alpha=\left(\frac{1}{r}-\frac{1}{p}\right)\left(\frac{1}{r}-\frac{1}{\bar{m}}\right)^{-1}$, $\bar{m}=\frac{nm}{n-m}$; *n is the dimension of Ω.*

The following two lemmas are similar to Lemmas 1 and 3 in [6].

Lemma 3. *If the conditions* (A) *are satisfied, then* $\|\boldsymbol{u}_h\|_{L_2}\leqslant E$, *where E is a definite constant.*

Lemma 4. *If the conditions* (A) *are satisfied, then* $\left\|\frac{\partial\boldsymbol{u}_h}{\partial x}\right\|_{L_2}\leqslant E_1$, *where E_1 is a definite constant.*

From Lemmas 1, 4 and 3, we have the following

Corollary 1. $\|\boldsymbol{u}_h\|_{L_\infty}\leqslant E_2$.

In order to estimate the error, we first introduce the elliptic projection of linear part P_h: $(\widetilde{H}^r)^N\to(U_h+iU_h)^N$ such that if $\boldsymbol{v}\in(\widetilde{H}^r)^N$, $P_h\boldsymbol{v}=(p_h^1v_1,\ \cdots,\ p_h^Nv_N)$, where $p_h^kv_k=p_h^kv_k^1+ip_h^kv_k^2$, $1\leqslant k\leqslant N$, satisfy

$$\begin{aligned} &A_k(v_k-p_h^kv_k,\ \chi)+(\alpha_k(\bar{v}_k-p_h^k\bar{v}_k)_x,\ \chi)+((K_{kk}+\widetilde{C})(v_k-p_h^kv_k),\ \chi) \\ &\quad=0,\quad\forall\chi\in U_h,\ k=1,\ 2,\ \cdots,\ N. \end{aligned} \tag{2.2}$$

Separating the real and imaginary parts, we have

$$\begin{aligned} &A_k(p_h^kv_k^1,\ \chi)+(\alpha_k^1(p_h^kv_k^1)_x+\alpha_k^2(p_h^kv_k^2)_x,\ \chi)+((K_{kk}+\widetilde{C})\,p_h^kv_k^1,\ \chi) \\ &\quad=A_k(v_k^1,\ \chi)+(\alpha_k^1v_{kx}^1+\alpha_k^2v_{kx}^2,\ \chi)+((K_{kk}+\widetilde{C})v_k^1,\ \chi), \\ &A_k(p_h^kv_k^2,\ \chi)+(\alpha_k^2(p_h^kv_k^1)_x-\alpha_k^1(p_h^kv_k^2)_x,\ \chi)+((K_{kk}+\widetilde{C})p_h^kv_k^2,\ \chi) \\ &\quad=A_k(v_k^2,\ \chi)+(\alpha_k^2v_{kx}^1-\alpha_k^1v_{kx}^2,\ \chi)+((K_{kk}+\widetilde{C})v_k^2,\ \chi), \end{aligned} \tag{2.3}$$

where $\widetilde{C}$ is a indeterminate and properly large positive number. We define the differential operators L_k as

$$L_kv_k=\begin{pmatrix} -\frac{\partial}{\partial x}\left(a_k\frac{\partial}{\partial x}\right)+\alpha_k^1\frac{\partial}{\partial x}+(K_{kk}+\widetilde{C}), & \alpha_k^2\frac{\partial}{\partial x} \\ \alpha_k^2\frac{\partial}{\partial x}, & -\frac{\partial}{\partial x}\left(a_k\frac{\partial}{\partial x}\right)-\alpha_k^1\frac{\partial}{\partial x}+(K_{kk}+\widetilde{C}) \end{pmatrix}\begin{pmatrix} v_k^1 \\ v_k^2 \end{pmatrix}$$

If we simply denote v_k^1 by $\tilde{v}_1$ and v_k^2 by $\tilde{v}_2$, we have

$$(L_k v_k,\ v_k)=\int_I\left\{a_k\left[\left(\frac{\partial \tilde{v}_1}{\partial x}\right)^2+\left(\frac{\partial \tilde{v}_2}{\partial x}\right)^2\right]+\frac{1}{2}(K_{kk}+\widetilde{C})\,(\tilde{v}_1^2+\tilde{v}_2^2)\right\}dx.$$

Notice

$$\left|\int_I (K_{kk}+\widetilde{C})\,(\tilde{v}_1^2+\tilde{v}_2^2)\,dx\right|\geqslant(\widetilde{C}-K)\,\|v_k\|_{L_2}^2.$$

When $\widetilde{C}$ is properly large we have

$$(L_k v_k,\ v_k)\geqslant m\|v_{kx}\|_{L_2}^2+(\widetilde{C}-K)\|v_k\|_{L_2}^2\geqslant m\|v_k\|_1^2,\quad k=1,\ 2,\cdots,\ N.$$

From this we know the elliptic projection $P_h\boldsymbol{v}$ exists uniquely.

Lemma 5. $\|\boldsymbol{v}-P_h\boldsymbol{v}\|_{L_2}\leqslant Ch^r\|\boldsymbol{v}\|_r,\ \forall \boldsymbol{v}\in \widetilde{H}^r)^N$.

Proof. We write

$$L\boldsymbol{v}=-\frac{\partial}{\partial x}\left(A(x)\,\frac{\partial \boldsymbol{v}}{\partial x}\right)+B\,\frac{\partial \overline{\boldsymbol{v}}}{\partial x}+(K_D+\widetilde{C}I)\boldsymbol{v}$$

where $K_D=\mathrm{diag}\ (K_{ii})$. Thus for any $\chi\in(U_h+iU_h)^N$ we have

$$\begin{aligned}
&m(\|(\boldsymbol{v}-P_h\boldsymbol{v})_x\|_{L_2}^2+\|\boldsymbol{v}-P_h\boldsymbol{v}\|_{L_2}^2)\\
&\leqslant|(L(\boldsymbol{v}-P_h\boldsymbol{v}),\ \boldsymbol{v}-P_h\boldsymbol{v})|\\
&=|A(\boldsymbol{v}-P_h\boldsymbol{v},\ \boldsymbol{v}-\chi)+(B(\overline{\boldsymbol{v}}-P_h\overline{\boldsymbol{v}})_x,\ \boldsymbol{v}-\chi)+((K_D+\widetilde{C}I)(\boldsymbol{v}-P_h\boldsymbol{v},\ \boldsymbol{v}-\chi)|\\
&\leqslant M_1(\|(\boldsymbol{v}-P_h\boldsymbol{v})_x\|_{L_2}+\|\boldsymbol{v}-P_h\boldsymbol{v}\|_{L_2})(\|(\boldsymbol{v}-\chi)_x\|_{L_2}+\|\boldsymbol{v}-\chi\|_{L_2}).
\end{aligned}$$

Eliminating $\|(\boldsymbol{v}-P_h\boldsymbol{v})_x\|_{L_2}+\|\boldsymbol{v}-P_h\boldsymbol{v}\|_{L_2}$ on both sides of the inequality, we obtain

$$\|(\boldsymbol{v}-P_h\boldsymbol{v})_x\|_{L_2}+\|\boldsymbol{v}-P_h\boldsymbol{v}\|_{L_2}\leqslant C\inf_{\chi\in(U_h+iU_h)^N}(\|(\boldsymbol{v}-\chi)_x\|_{L_2}+\|\boldsymbol{v}-\chi\|_{L_2}).$$

According to the interpolation theory of Sobolev space [10], we know

$$\|(\boldsymbol{v}-P_h\boldsymbol{v})_x\|_{L_2}+\|\boldsymbol{v}-P_h\boldsymbol{v}\|_{L_2}\leqslant Ch^{r-1}\|\boldsymbol{v}\|_r.$$

On the other hand, the conjugate of operator L_k is

$$L_k^*=\begin{pmatrix}-\dfrac{\partial}{\partial x}\left(a_k\dfrac{\partial}{\partial x}\right)-\alpha_k^1\dfrac{\partial}{\partial x}+(K_{kk}+\widetilde{C}), & -\alpha_k^2\dfrac{\partial}{\partial x}\\ -\alpha_k^2\dfrac{\partial}{\partial x}, & -\dfrac{\partial}{\partial x}\left(a_k\dfrac{\partial}{\partial x}\right)+\alpha_k^1\dfrac{\partial}{\partial x}+(K_{kk}+\widetilde{C})\end{pmatrix}.$$

The characteristic polynomial of L_k^* is

$$\zeta_k(x,\ \xi)=\begin{vmatrix}-a_k(x)(i\xi)^2, & -\alpha_k^2(i\xi)\\ -\alpha_k^2(i\xi), & -a_k(x)(i\xi)^2\end{vmatrix}=(\alpha_k^2)^2\xi^2+a_k^2(x)\xi^4\neq0,\quad\forall\xi\neq0. \tag{2.4}$$

So L_k^* is an elliptic operator. According to the results of [11], for any $\varphi_1,\ \varphi_2\in\widetilde{H}^m$, the solutions of system $L_k^*\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}=\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}$, $k=1,\ 2,\ \cdots,\ N$, have the following estimate

$$\sum_{i=1}^2\|\psi_i\|_{m+2}^2\leqslant C\sum_{i=1}^2\|\varphi_i\|_m^2,\ m\geqslant0.$$

Thus from

$$\left(\begin{pmatrix}v_k^1-p_h^k v_k^1\\ v_k^2-p_h^k v_k^2\end{pmatrix},\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}\right)=\left(\begin{pmatrix}v_k^1-p_h^k v_h^1\\ v_k^2-p_h^k v_k^2\end{pmatrix},\ L_k^*\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}\right)$$

$$-\int_I \Big\{a_k\Big[\frac{\partial(v_k^1-p_h^k v_k^1)}{\partial x}\cdot\frac{\partial(\psi_1-\chi_1)}{\partial x}+\frac{\partial(v_k^2-p_h^k v_k^2)}{\partial x}\cdot\frac{\partial(\psi_2-\chi_2)}{\partial x}\Big]$$
$$+\Big[\alpha_k^1\frac{\partial(v_k^2-p_h^k v_k^2)}{\partial x}+(K_{kk}+\widetilde{C})(v_k^1-p_h^k v_k^1)+\alpha_k^2\frac{\partial(v_k^2-p_h^k v_k^2)}{\partial x}\Big](\psi_1-\chi_1)$$
$$+\Big[-\alpha_k^1\frac{\partial(v_k^2-p_h^k v_k^2)}{\partial x}+(K_{kk}+\widetilde{C})(v_k^2-p_h^k v_k^2)+\alpha_k^2\frac{\partial(v_k^1-p_h^k v_k^1)}{\partial x}\Big](\psi_2-\chi_2)\Big\}dx$$

we obtain for any $\chi_1,\ \chi_2\in U_h$

$$\left|\left(\begin{pmatrix}v_h^1-p_h^k v_h^1\\ v_k^2-p_h^k v_k^2\end{pmatrix},\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}\right)\right|$$
$$\leqslant M_1\|(v_k^1-p_h^k v_k^1)_x\|_{L_2}\|(\psi_1-\chi_1)_x\|_{L_2}+M_1\|(v_k^2-p_h^k v_k^2)_x\|_{L_2}\|(\psi_2-\chi_2)_x\|_{L_2}$$
$$+M_2\|(v_k^1-p_h^k v_k^1)_x\|_{L_2}\|\psi_1-\chi_1\|_{L_2}+M_3\|(v_k^1-p_h^k v_k^1)_x\|_{L_2}\|\psi_2-\chi_2\|_{L_2}$$
$$+(K+\widetilde{C})[\|v_k^1-p_h^k v_k^1\|_{L_2}\|\psi_1-\chi_1\|_{L_2}+\|v_k^2-p_h^k v_k^2\|_{L_2}\|\psi_2-\chi_2\|_{L_2}]$$
$$+M_2\|(v_k^2-p_h^k v_k^2)_x\|_{L_2}\|\psi_2-\chi_2\|_{L_2}+M_3\|(v_k^2-p_h^k v_k^2)_x\|_{L_2}\|\psi_1-\chi_1\|_{L_2}.$$

Hence

$$\left|\left(\begin{pmatrix}v_k^1-p_h^k v_k^1\\ v_k^2-p_h^k v_k^2\end{pmatrix},\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}\right)\right|\leqslant Ch^r\|\boldsymbol{v}\|_r(\|\varphi_1\|_{L_2}+\|\varphi_2\|_{L_2}),$$

i. e.

$$\|\boldsymbol{v}-P_h\boldsymbol{v}\|_{L_2}^2\leqslant\sum_{k=1}^N\|v_k^1-p_h^k v_k^1\|_{L_2}^2+\sum_{k=1}^N\|v_k^2-p_h^k v_k^2\|_{L_2}^2\leqslant Ch^{2r}\|\boldsymbol{v}\|_r^2.$$

Now we estimate the error of the semi-discrete solution.

Theorem 1. *Assume that* $\boldsymbol{u}(x,t)$ *is the solution of* (1.1) *and* $\boldsymbol{u}(x,t)$, $\boldsymbol{u}_t(x,t)\in L^\infty(0,T;\widetilde{H}^r)$, $r>1$, $U_h\subset\widetilde{H}^2$. If $\|\varepsilon_h(0)\|_{L_2}=\|\boldsymbol{u}(0)-\boldsymbol{u}_h(0)\|_{L_2}\leqslant Ch^r$, then

$$\|\boldsymbol{u}(t)-\boldsymbol{u}_h(t)\|_{L_2}\leqslant Ch^r,\quad 0\leqslant t\leqslant T.$$

Proof. Let $\varepsilon_l=u_l-u_{hl}=u_l-p_h^l u_l+p_h^l u_l-u_{hl}=\mathscr{D}_l-\rho_l$. Then $\rho_l\in U_h+iU_h$ and satisfy

$$i\Big(\frac{\partial\rho_l}{\partial t},\chi\Big)+A_l(\rho_l,\chi)+(\alpha_l\bar{\rho}_{lx},\chi)+(\beta(x)q(|\boldsymbol{u}_h|^2)u_{hl},\chi)$$
$$-(\beta(x)q(|P_h\boldsymbol{u}|^2)p_h^l u_l,\ \chi)+\sum_{j=1}^N(K_{lj}\rho_j,\ \chi)$$
$$=i\Big(\frac{\partial\mathscr{D}_l}{\partial t},\chi\Big)+A_l(\mathscr{D}_l,\ \chi)+(\alpha_l\overline{\mathscr{D}}_{lx},\ \chi)+(\beta(x)q(|\boldsymbol{u}|^2)u_l,\ \chi)$$
$$-(\beta(x)q(|P_h\boldsymbol{u}|^2)p_h^l u_l,\ \chi)+\sum_{j=1}^N(K_{lj}\mathscr{D}_j,\ \chi),\quad\forall\chi\in U_h+iU_h.\tag{2.5}$$

We put $\chi=\rho_l$, take the imaginary part on both sides of equality (2.5) and make the following estimations

$$|(\alpha_l\bar{\rho}_{lx},\ \rho_l)|=\frac{1}{2}\left|\int_I\alpha_l\frac{\partial}{\partial x}(\bar{\rho}_l)^2dx\right|=0,$$

$$|(\beta(x)[q(|\boldsymbol{u}_h|^2)u_{hl}-q(|P_h\boldsymbol{u}|^2)p_h^l u_l],\ \rho_l)|$$
$$\leqslant|(\beta(x)[q(|\boldsymbol{u}_h|^2)-q(|P_h\boldsymbol{u}|^2)]u_{hl},\ \rho_l)|+|(\beta(x)q(|P_h\boldsymbol{u}|^2)\rho_l,\ \rho_l)|$$
$$\leqslant K_1\|q'(s)\|_{L_\infty}(\|\boldsymbol{u}_h\|_{L_\infty}^2+\|\boldsymbol{u}_h\|_{L_\infty}\|P_h\boldsymbol{u}\|_{L_\infty})\|\rho\|_{L_2}\|\rho_l\|_{L_2}+K_2\|\rho_l\|_{L_2}^2\|P_h\boldsymbol{u}\|_{L_\infty}^2,$$

$$\left|\Big(\frac{\partial\mathscr{D}_l}{\partial t},\ \rho_l\Big)\right|\leqslant\frac{1}{2}\Big(\Big\|\frac{\partial\mathscr{D}_l}{\partial t}\Big\|_{L_2}^2+\|\rho_l\|_{L_2}^2\Big).$$

For $A_l(\mathscr{D}_l, \rho_l)$, according to property of the elliptic projection, we have

$$A_l(\mathscr{D}_l, \rho_l) = -(\alpha_l(\overline{u_l - p_h^l u_l})_x, \rho_l) - ((K_u + \widetilde{C})(u_l - p_h^l u_l), \rho_l).$$

Thus

$$A_l(\mathscr{D}_l, \rho_l) + (\alpha_l \overline{\mathscr{D}}_{lx}, \rho_l) + \sum_{j=1}^{N}(K_{lj}\mathscr{D}_j, \rho_l) = \sum_{j\neq l}(K_{lj}\mathscr{D}_j, \rho_l) - \widetilde{C}(\mathscr{D}_l, \rho_l).$$

Hence

$$|A_l(\mathscr{D}_l, \rho_l) + (\alpha_l \overline{\mathscr{D}}_{lx}, \rho_l) + \sum_{j=1}^{N}(K_{lj}\mathscr{D}_j, \rho_l)| \leqslant K_3 \sum_{j=1}^{N} \|\mathscr{D}_j\|_{L_2} \|\rho_l\|_{L_2}.$$

Similarly, we have

$$\begin{aligned} &|(\beta(x)[q(|\boldsymbol{u}|^2)u_l - q(|P_h\boldsymbol{u}|^2)p_h^l u_l], \rho_l)| \\ &\leqslant K_4 \|q'(s)\|_{L_\infty} [\|\boldsymbol{u}\|_{L_\infty}^2 + \|\boldsymbol{u}\|_{L_\infty} \|P_h\boldsymbol{u}\|_{L_\infty}] \cdot \sum_{j=1}^{N} \|\mathscr{D}_j\|_{L_2} \|\rho_l\|_{L_2} \\ &\quad + K_5 \|P_h\boldsymbol{u}\|_{L_\infty}^2 \|\mathscr{D}_l\|_{L_2} \|\rho_l\|_{L_2}. \end{aligned}$$

Finally,

$$\left|\sum_{j=1}^{N}(K_{lj}\rho_j, \rho_l)\right| \leqslant K_6 \sum_{j=1}^{N} \|\rho_j\|_{L_2} \|\rho_l\|_{L_2},$$

From the Corollary of Lemma 3 in [6] we obtain $\|\boldsymbol{u}\|_{L_\infty} \leqslant E$ and using Sobolev estimation we have $\|P_h\boldsymbol{u}\|_{L_\infty} \leqslant E_1$ uniformly for h. Now sum up both sides of (2.5) for l from 1 to N and substitute the above estimates into (2.5). After simplification we have

$$\frac{d}{dt}\|\boldsymbol{\rho}\|_{L_2}^2 \leqslant B_1 \|\boldsymbol{\rho}\|_{L_2}^2 + B_2\left[\|\mathscr{D}\|_{L_2}^2 + \left\|\frac{\partial \mathscr{D}}{\partial t}\right\|_{L_2}^2\right].$$

Using the Gronwall inequality, we obtain

$$\|\boldsymbol{\rho}(t)\|_{L_2}^2 \leqslant e^{B_1 T}\left\{\|\boldsymbol{\rho}(0)\|_{L_2}^2 + B_2 T \sup_{0\leqslant t\leqslant T}\left[e^{-B_1 t}\left(\|\mathscr{D}(t)\|_{L_2}^2 + \left\|\frac{\partial \mathscr{D}(t)}{\partial t}\right\|_{L_2}^2\right)\right]\right\}, \quad 0 \leqslant t \leqslant T.$$

From $\|\boldsymbol{u}_h(0) - P_h\boldsymbol{u}(0)\|_{L_2} \leqslant \|\boldsymbol{u}(0) - P_h\boldsymbol{u}(0)\|_{L_2} + \|\boldsymbol{u}(0) - \boldsymbol{u}_h(0)\|_{L_2} \leqslant Ch^r$ and $\|\boldsymbol{u}(t) - \boldsymbol{u}_h(t)\|_{L_2} \leqslant \|\mathscr{D}\|_{L_2} + \|\boldsymbol{\rho}\|_{L_2}$, by Lemma 5 and the conditions of Theorem 1, the theorem is established.

§ 3. The Fully Discrete Finite Element Method and Its Error Estimation

Now we consider the fully discrete scheme of the finite element method. Suppose $q(s) = s$. Using the improved Euler method, the fully discrete scheme is as follows:

$$\left\{\begin{aligned} &i\left(\frac{U_l^{n+1} - U_l^n}{k}, \chi\right) + \frac{1}{2}A_l(U_l^{n+1} + U_l^n, \chi) + \frac{1}{2}(\alpha_l(\overline{U}_l^{n+1} + \overline{U}_l^n)_x, \chi) \\ &\quad + \frac{1}{4}(\beta(x)(|U^{n+1}|^2 + |U^n|^2)(U_l^{n+1} + U_l^n), \chi) + \frac{1}{2}\sum_{j=1}^{N}(K_{lj}(\cdot, t^{n+\frac{1}{2}})(U_j^{n+1} + U_j^n), \chi) \\ &= (G_l(\cdot, t^{n+\frac{1}{2}}), \chi), \\ &U^0 = \boldsymbol{u}_h(0), \ U_l^n(x + 2\pi) = U_l^n(x), \ -\infty < x < +\infty, \end{aligned}\right. \tag{3.1}$$

$$0\leqslant n\leqslant\left[\frac{T}{k}\right]-1,\ \forall\chi\in U_h+iU_h,\ l=1,\ 2,\ \cdots,\ N.$$

Lemma 6. $\|U^s\|_{L_2}\leqslant E_0,\ 0\leqslant s\leqslant\left[\frac{T}{k}\right]$; *$E_0$ is a definite constant.*

Proof. Set $\chi=U_l^{n+1}+U_l^n$ in equality (3.1). We obtain

$$i\left(\frac{U_l^{n+1}-U_l^n}{k},\ U_l^{n+1}+U_l^n\right)+\frac{1}{2}A_l(U_l^{n+1}+U_l^n,U_l^{n+1}+U_l^n)$$
$$+\frac{1}{2}(\alpha_l(\overline{U}_l^{n+1}+\overline{U}_l^n)_x,\ U_l^{n+1}+U_l^n)+\frac{1}{4}(\beta(x)(|U^{n+1}|^2+|U^n|^2)(U_l^{n+1}+U_l^n),$$
$$U_l^{n+1}+U_l^n)+\frac{1}{2}\sum_{j=1}^{N}(K_{lj}(\cdot,\ t^{n+\frac{1}{2}})(U_j^{n+1}+U_j^n),\ U_l^{n+1}+U_l^n)$$
$$=(G_l(\cdot,\ t^{n+\frac{1}{2}}),\ U_l^{n+1}+U_l^n),\quad l=1,\ 2,\ \cdots,\ N. \tag{3.2}$$

Notice

$$\operatorname{Im} i\left(\frac{U_l^{n+1}-U_l^n}{k},U_l^{n+1}+U_l^n\right)=\frac{1}{k}\int_I|U_l^{n+1}|^2dx-\frac{1}{k}\int_I|U_l^n|^2dx,$$
$$A_l(U_l^{n+1}+U_l^n,\ U_l^{n+1}+U_l^n)>0,$$
$$\frac{1}{2}(\alpha_l(\overline{U}_l^{n+1}+\overline{U}_l^n)_x,\ U_l^{n+1}+U_l^n)=\frac{1}{2}\int_I\alpha_l\frac{\partial}{\partial x}(\overline{U}_l^{n+1}+\overline{U}_l^n)^2dx=0,$$
$$\operatorname{Im}\left(\frac{1}{4}\beta(x)(|U^{n+1}|^2+|U^n|^2)(U_l^{n+1}+U_l^n),\ U_l^{n+1}+U_l^n\right)=0,$$

and because $K_{lj}=K_{jl}$, we have

$$\operatorname{Im}\frac{1}{2}\sum_{l=1}^{N}\sum_{j=1}^{N}(K_{lj}(\cdot,\ t^{n+\frac{1}{2}})(U_j^{n+1}+U_j^n),\ U_l^{n+1}+U_l^n)=0,$$
$$|(G_l(\cdot,\ t^{n+\frac{1}{2}}),\ U_l^{n+1}+U_l^n)|\leqslant M_1+M_2(\|U_l^{n+1}\|_{L_2}^2+\|U_l^n\|_{L_2}^2).$$

We take the imaginary part on both sides of equality (3.2) and sum up for l from 1 to N. Using the above estimates we obtain

$$\|U^{n+1}\|_{L_2}^2\leqslant\|U^n\|_{L_2}^2+kM_1+kM_2(\|U^{n+1}\|_{L_2}^2+\|U^n\|_{L_2}^2).$$

When k is small enough, we obtain the result of the lemma.

Lemma 7. $\|U_x^s\|_{L_2}\leqslant E_1,\ 0\leqslant s\leqslant\left[\frac{T}{k}\right]$; *$E_1$ is a definite constant.*

Proof. Set $\chi=U_l^{n+1}-U_l^n$ in equality (3.1) and we obtain

$$i\left(\frac{U_l^{n+1}-U_l^n}{k},\ U_l^{n+1}-U_l^n\right)+\frac{1}{2}A_l(U_l^{n+1}+U_l^n,\ U_l^{n+1}-U_l^n)$$
$$+\frac{1}{2}(\alpha_l(\overline{U}_l^{n+1}+\overline{U}_l^n)_x,\ U_l^{n+1}-U_l^n)$$
$$+\frac{1}{4}(\beta(x)(|U^{n+1}|^2+|U^n|^2)(U_l^{n+1}+U_l^n),U_l^{n+1}-U_l^n)$$
$$+\frac{1}{2}\sum_{j=1}^{N}(K_{lj}(\cdot,\ t^{n+\frac{1}{2}})(U_j^{n+1}+U_j^n),U_l^{n+1}-U_l^n)$$
$$=(G_l(\cdot,t^{n+\frac{1}{2}}),\ U_l^{n+1}-U_l^n),\quad l=1,\ 2,\ \cdots,\ N. \tag{3.3}$$

It is obvious that

$$\operatorname{Re} i\left(\frac{U_l^{n+1}-U_l^n}{k},\ U_l^{n+1}-U_l^n\right)=0,$$

$$\operatorname{Re}\frac{1}{2}\int_I a_l\,\frac{\partial(U_l^{n+1}+U_l^n)}{\partial x}\,\frac{\partial(\overline{U}_l^{n+1}-\overline{U}_l^n)}{\partial x}\,dx=\frac{1}{2}\int_I a_l(x)\left[\left|\frac{\partial U_l^{n+1}}{\partial x}\right|^2-\left|\frac{\partial U_l^n}{\partial x}\right|^2\right]dx.$$

In order to estimate $\frac{1}{2}(\alpha_l(\overline{U}_l^{n+1}+\overline{U}_l^n)_x,\ U_l^{n+1}-U_l^n)$, we take $\chi=\frac{1}{2}\alpha_l(\overline{U}_l^{n+1}+\overline{U}_l^n)_x$ in equality (3.1) and obtain

$$\begin{aligned}
&\left(U_l^{n+1}-U_l^n,\ \frac{1}{2}\alpha_l(\overline{U}_l^{n+1}+\overline{U}_l^n)_x\right)\\
&=ik\Big\{\frac{1}{4}A_l(U_l^{n+1}+U_l^n,\ \alpha_l(\overline{U}_l^{n+1}+\overline{U}_l^n)_x)+\frac{1}{4}(\alpha_l(\overline{U}_l^{n+1}+\overline{U}_l^n)_x,\ \alpha_l(\overline{U}_l^{n+1}+\overline{U}_l^n)_x)\\
&\quad+\frac{1}{16}(\beta(x)(|U^{n+1}|^2+|U^n|^2)(U_l^{n+1}+U_l^n),\ \alpha_l(\overline{U}_l^{n+1}+\overline{U}_l^n)_x)\\
&\quad+\frac{1}{4}\sum_{j=1}^{N}(K_{lj}(\cdot,\ t^{n+\frac{1}{2}})(U_l^{n+1}+U_l^n),\ \alpha_l(\overline{U}_l^{n+1}+\overline{U}_l^n)_x)\\
&\quad-\frac{1}{2}(G_l(\cdot,\ t^{n+\frac{1}{2}}),\ \alpha_l(\overline{U}_l^{n+1}+\overline{U}_l^n)_x)\Big\}.
\end{aligned}\tag{3.4}$$

We estimate each term of the above equality as follows

$$\begin{aligned}
&\frac{1}{4}ik\,A_l(U_l^{n+1}+U_l^n,\ \alpha_l(\overline{U}_l^{n+1}+\overline{U}_l^n)_x)\\
&=\frac{1}{4}ik\int_I a_l(x)\overline{\alpha}_l(U_l^{n+1}+U_l^n)_x(U_l^{n+1}+U_l^n)_x\,dx\\
&=\frac{1}{8}ik\int_I a_l(x)\overline{\alpha}_l\frac{\partial}{\partial x}[(U_l^{n+1}+U_l^n)_x]^2\,dx\\
&=-\frac{1}{8}ik\int_I \overline{\alpha}_l a_{lx}(x)(U_{lx}^{n+1}+U_{lx}^n)^2\,dx,
\end{aligned}$$

$$\operatorname{Re}\frac{ik}{4}\int_I|\alpha_l|^2|U_{lx}^{n+1}+U_{lx}^n|^2\,dx=0,$$

$$\begin{aligned}
&\left|\frac{1}{8}ik\int_I\overline{\alpha}_l\beta(x)(|U^{n+1}|^2+|U^n|^2)(U_l^{n+1}+U_l^n)(U_l^{n+1}+U_l^n)_x\,dx\right|\\
&\leqslant kK_1\sum_{j=1}^{N}\int_I(|U_j^{n+1}|^2+|U_j^n|^2)(|U_l^{n+1}|\,|U_{lx}^{n+1}|+|U_{lx}^n|\,|U_l^{n+1}|\\
&\quad+|U_{lx}^{n+1}|\,|U_l^n|+|U_{lx}^n|\,|U_l^n|)\,dx.
\end{aligned}$$

According to the Hölder inequality, Lemmas 2 and 6, we have

$$\begin{aligned}
&\int_I|U_j^{n+1}|^2|U_{lx}^{n+1}|\,|U_l^{n+1}|\,dx\\
&\leqslant\left(\int_I|U_{lx}^{n+1}|^2dx\right)^{\frac{1}{2}}\left(\int_I|U_j^{n+1}|^6dx\right)^{\frac{1}{3}}\left(\int_I|U_l^{n+1}|^6dx\right)^{\frac{1}{6}}\\
&\leqslant\int_I|U_{lx}^{n+1}|^2dx+\int_I|U_j^{n+1}|^6dx+\int_I|U_l^{n+1}|^6dx\\
&\leqslant K_2(\|U_{lx}^{n+1}\|_{L_2}^2+\|U_{jx}^{n+1}\|_{L_2}^2)+K_3.
\end{aligned}$$

For other terms we have similar estimates. Hence

$$\left|\frac{1}{8}ik\int_I \bar{a}_l\beta(x)(|U^{n+1}|^2+|U^n|^2)(U_l^{n+1}+U_l^n)(\bar{U}_l^{n+1}+\bar{U}_l^n)_x dx\right|$$
$$\leqslant kK_4(\|U_x^{n+1}\|_{L_2}^2+\|U_x^n\|_{L_2}^2)+kK_5.$$

In addition

$$\left|\frac{1}{4}ik\int_I \bar{a}_l\sum_{j=1}^N K_{lj}(x,t^{n+\frac{1}{2}})(U_j^{n+1}+U_j^n)(\bar{U}_{lx}^{n+1}+\bar{U}_{lx}^n)dx\right|$$
$$\leqslant kK_6(\|U_x^{n+1}\|_{L_2}^2+\|U_x^n\|_{L_2}^2)+kK_7,$$

$$\left|\frac{1}{2}ik\int_I \bar{a}_l(\bar{U}_{lx}^{n+1}+\bar{U}_{lx}^n)G_l(x,t^{n+\frac{1}{2}})dx\right|\leqslant kK_8(\|U_x^{n+1}\|_{L_2}^2+\|U_x^n\|_{L_2}^2)+kK_9.$$

Using the above estimates in equality (3.4), we obtain

$$\sum_{l=1}^N\left|\frac{1}{2}(a_l(\bar{U}_{lx}^{n+1}+\bar{U}_{lx}^n),\ U_l^{n+1}-U_l^n)\right|\leqslant k[K_{10}(\|U_x^{n+1}\|_{L_2}^2+\|U_x^n\|_{L_2}^2)+K_{11}].$$

Suppose $K_{lj}^{n+\frac{1}{2}}=K_{lj}(x,t^{n+\frac{1}{2}})$, $K_{lj}^{-\frac{1}{2}}=0$, $G_l(x,t^{-\frac{1}{2}})=0$. We have

$$\mathrm{Re}\sum_{l=1}^N\sum_{j=1}^N\frac{1}{2}(K_{lj}(\cdot,t^{n+\frac{1}{2}})(U_j^{n+1}+U_j^n),\ U_l^{n+1}-U_l^n)$$
$$=\frac{1}{2}\mathrm{Re}\int_I\sum_{l=1}^N\sum_{j=1}^N K_{lj}^{n+\frac{1}{2}}U_j^{n+1}\bar{U}_l^{n+1}dx-\frac{1}{2}\mathrm{Re}\int_I\sum_{l=1}^N\sum_{j=1}^N K_{lj}^{n-\frac{1}{2}}U_j^n\bar{U}_l^n dx$$
$$+\frac{1}{2}\mathrm{Re}\int_I\sum_{l=1}^N\sum_{j=1}^N(K_{lj}^{n-\frac{1}{2}}-K_{lj}^{n+\frac{1}{2}})U_j^n\bar{U}_l^n dx.$$

But

$$\left|\frac{1}{2}\int_I\sum_{l=1}^N\sum_{j=1}^N(K_{lj}^{n-\frac{1}{2}}-K_{lj}^{n+\frac{1}{2}})U_j^n\bar{U}_l^n dx\right|\leqslant kK_{12}\|U^n\|_{L_2}^2,$$

$$\mathrm{Re}\int_I G_l(x,t^{n+\frac{1}{2}})(\bar{U}_l^{n+1}-\bar{U}_l^n)dx$$
$$=\mathrm{Re}\int_I G_l(x,t^{n+\frac{1}{2}})\bar{U}_l^{n+1}dx-\mathrm{Re}\int_I G_l(x,t^{n-\frac{1}{2}})\bar{U}_l^n dx$$
$$+\mathrm{Re}\int_I[G_l(x,t^{n-\frac{1}{2}})-G_l(x,t^{n+\frac{1}{2}})]\bar{U}_l^n dx.$$

Moreover, notice

$$\mathrm{Re}\sum_{l=1}^N\frac{1}{4}(\beta(x)(|U^{n+1}|^2+|U^n|^2)(U_l^{n+1}+U_l^n),\ U_l^{n+1}-U_l^n)$$
$$=\frac{1}{4}\int_I\beta(x)(|U^{n+1}|^4-|U^n|^4)dx.$$

We take the real part on both sides of equality (3.3) and sum up for l from 1 to N. We then obtain

$$\frac{1}{2}\sum_{l=1}^N\int_I a_l(x)\left[\left|\frac{\partial U_l^{n+1}}{\partial x}\right|^2-\left|\frac{\partial U_l^n}{\partial x}\right|^2\right]dx+\frac{1}{4}\int_I\beta(x)(|U^{n+1}|^4-|U^n|^4)dx$$
$$+\frac{1}{2}\mathrm{Re}\int_I\sum_{l=1}^N\sum_{j=1}^N K_{lj}(x,t^{n+\frac{1}{2}})U_j^{n+1}\bar{U}_l^{n+1}dx-\frac{1}{2}\mathrm{Re}\int_I\sum_{l=1}^N\sum_{j=1}^N K_{lj}(x,t^{n-\frac{1}{2}})U_j^n\bar{U}_l^n dx$$
$$\leqslant\sum_{l=1}^N\mathrm{Re}\int_I G_l(x,t^{n+\frac{1}{2}})\bar{U}_l^{n+1}dx-\sum_{l=1}^N\mathrm{Re}\int_I G_l(x,t^{n-\frac{1}{2}})\bar{U}_l^n dx$$

$$+\sum_{l=1}^{N}\mathrm{Re}\int_I [G_l(x,\ t^{n-\frac{1}{2}})-G_l(x,\ t^{n+\frac{1}{2}})]\overline{U}_l^n dx+kK_{13}(\|U_x^{n+1}\|_{L_2}^2+\|U_x^n\|_{L_2}^2)+K_{14}k.$$

Again summing up for n from 0 to $H-1$ on both sides of the above equality, we obtain

$$\begin{aligned}\frac{1}{2}\sum_{l=1}^{N}\int_I a_l(x)\left|\frac{\partial U_l^H}{\partial x}\right|^2 dx+\frac{1}{4}\int_I\beta(x)|U^H|^4dx+\frac{1}{2}\mathrm{Re}\int_I\sum_{l=1}^{N}\sum_{j=1}^{N}K_{lj}(x,\ t^{H-\frac{1}{2}})U_j^H\overline{U}_l^H dx\\
\leqslant\frac{1}{2}\sum_{l=1}^{N}\int_I a_l(x)\left|\frac{\partial U_l^0}{\partial x}\right|^2 dx+\frac{1}{4}\int_I\beta(x)|U^0|^4dx+\sum_{l=1}^{N}\mathrm{Re}\int_I G_l(x,\ t^{H-\frac{1}{2}})\overline{U}_l^H dx\\
+\sum_{n=1}^{H-1}\mathrm{Re}\int_I\sum_{l=1}^{N}[G_l(x,\ t^{n-\frac{1}{2}})-G_l(x,\ t^{n+\frac{1}{2}})]\overline{U}_l^n dx-\sum_{l=1}^{N}\mathrm{Re}\int_I G_l(x,\ t^{\frac{1}{2}})\overline{U}_l^0 dx\\
+kK_{13}\sum_{n=0}^{H-1}(\|U_x^{n+1}\|_{L_2}^2+\|U_x^n\|_{L_2}^2)+K_{15}.\end{aligned}\tag{3.5}$$

Notice the conditions (A) and Lemma 2 and we have

$$\begin{aligned}\left|\frac{1}{4}\int_I\beta(x)|U^H|^4dx\right|&\leqslant\frac{B_0}{4}\int_I|U^H|^4dx\leqslant B^*\|U^H\|_1\|U^H\|_{L_2}^3\\
&\leqslant B^*(\|U_x^H\|_{L_2}+\|U^H\|_{L_2})\|U^H\|_{L_2}^3\\
&\leqslant\varepsilon\|U_x^H\|_{L_2}^2+C(\varepsilon)\|U^H\|_{L_2}^6+B^*\|U^H\|_{L_2}^4.\end{aligned}$$

Substituting the above estimates into (3.5) we obtain

$$\left(\frac{1}{2}m-\varepsilon-kK_{13}\right)\|U_x^H\|_{L_2}^2\leqslant K_{16}+kK_{17}\sum_{n=0}^{H-1}\|U_x^n\|_{L_2}^2.$$

According to the Gronwall inequality, when k and ε are both small enough we have

$$\|U_x^s\|_{L_2}\leqslant E_1,\quad 0\leqslant s\leqslant\left[\frac{T}{k}\right].$$

Using Lemma 1, we immediately obtain

Corollary 2. $\|U^s\|_{L_\infty}\leqslant C,\ 0\leqslant s\leqslant\left[\frac{T}{k}\right]$, where C is a definite constant.

Finally we estimate the error of the fully discrete finite element solution. At $t=\left(n+\frac{1}{2}\right)k$ equations (1.2) can be written as

$$\begin{aligned}&i\left(\frac{u_l^{n+1}-u_l^n}{k},\ \chi\right)+\frac{1}{2}A_l(u_l^{n+1}+u_l^n,\ \chi)+\frac{1}{2}(\alpha_l(\overline{u}_l^{n+1}+\overline{u}_l^n)_x,\ \chi)\\
&\quad+\frac{1}{4}(\beta(x)(|u^{n+1}|^2+|u^n|^2)(u_l^{n+1}+u_l^n),\ \chi)+\frac{1}{2}\sum_{j=1}^{N}(K_{lj}(\cdot,\ t^{n+\frac{1}{2}})(u_j^{n+1}+u_j^n),\ \chi)\\
&=(G_l(\cdot,\ t^{n+\frac{1}{2}}),\ \chi)+(O(k^2),\ \chi),\quad\forall\chi\in U_h+iU_h,\ l=1,\ 2,\ \cdots,\ N.\end{aligned}\tag{3.6}$$

Let $U_l^n-u_l^n=U_l^n-p_h^l u_l^n-(u_l^n-p_h^l u_l^n)=\xi_l^n-\eta_l^n,\ l=1,\ 2,\ \cdots,\ N$. From (3.1) and (3.6), ξ_l^n satisfy

$$\begin{aligned}&i\left(\frac{\xi_l^{n+1}-\xi_l^n}{k},\ \chi\right)+\frac{1}{2}A_l(\xi_l^{n+1}+\xi_l^n,\ \chi)+\frac{1}{2}(\alpha_l(\overline{\xi}_l^{n+1}+\overline{\xi}_l^n)_x,\ \chi)\\
&\quad+\frac{1}{4}(\beta(x)(|U^{n+1}|^2+|U^n|^2)(U_l^{n+1}+U_l^n),\ \chi)\end{aligned}$$

$$-\frac{1}{4}(\beta(x)(|P_h\boldsymbol{u}^{n+1}|^2+|P_h\boldsymbol{u}^n|^2)(p_h^l u_l^{n+1}+p_h^l u_l^n),\ \chi)$$
$$+\frac{1}{2}\sum_{j=1}^{N}(K_{lj}(\cdot,\ t^{n+\frac{1}{2}})(\xi_j^{n+1}+\xi_j^n),\ \chi)$$
$$=i\left(\frac{\eta_l^{n+1}-\eta_l^n}{k},\ \chi\right)+\frac{1}{2}A_l(\eta_l^{n+1}+\eta_l^n,\ \chi)+\frac{1}{2}(\alpha_l(\bar\eta_l^{n+1}+\bar\eta_l^n)_x,\ \chi)$$
$$+\frac{1}{4}(\beta(x)(|\boldsymbol{u}^{n+1}|^2+|\boldsymbol{u}^n|^2)(u_l^{n+1}+u_l^n),\ \chi)$$
$$-\frac{1}{4}(\beta(x)(|P_h\boldsymbol{u}^{n+1}|^2+|P_h\boldsymbol{u}^n|^2)(p_h^l u_l^{n+1}+p_h^l u_l^n),\ \chi)$$
$$+\frac{1}{2}\sum_{j=1}^{N}(K_{lj}(\cdot,\ t^{n+\frac{1}{2}})(\eta_j^{n+1}+\eta_j^n),\ \chi)+(O(k^2),\chi),\ \forall x\in U_h+iU_h,\ l=1,\ 2,\cdots,\ N. \tag{3.7}$$

Let $\chi=\xi_l^{n+1}+\xi_l^n$, $l=1,\ 2,\ \cdots,\ N$ Notice

$$\operatorname{Im} i\left(\frac{\xi_l^{n+1}-\xi_l^n}{k},\ \xi_l^{n+1}+\xi_l^n\right)=\frac{1}{k}\int_I|\xi_l^{n+1}|^2dx-\frac{1}{k}\int_I|\xi_l^n|^2dx$$
$$=\frac{1}{k}(\|\xi_l^{n+1}\|_{L_2}^2-\|\xi_l^n\|_{L_2}^2),$$
$$A_l(\xi_l^{n+1}+\xi_l^n,\ \xi_l^{n+1}+\xi_l^n)>0,$$
$$\frac{1}{2}(\alpha_l(\bar\xi_l^{n+1}+\bar\xi_l^n)_x,\ \xi_l^{n+1}+\xi_l^n)=0.$$

Because

$$(|\boldsymbol{U}^{n+1}|^2+|\boldsymbol{U}^n|^2)(U_l^{n+1}+U_l^n)-(|P_h\boldsymbol{u}^{n+1}|^2+|P_h\boldsymbol{u}^n|^2)(p_h^l u_l^{n+1}+p_h^l u_l^n)$$
$$=(|\boldsymbol{U}^{n+1}|^2+|\boldsymbol{U}^n|^2)(\xi_l^{n+1}+\xi_l^n)+[(|\boldsymbol{U}^{n+1}|^2-|P_h\boldsymbol{u}^{n+1}|^2)$$
$$+(|\boldsymbol{U}^n|^2-|P_h\boldsymbol{u}^n|^2)](p_h^l u_l^{n+1}+p_h^l u_l^n)$$
$$=(|\boldsymbol{U}^{n+1}|^2+|\boldsymbol{U}^n|^2)(\xi_l^{n+1}+\xi_l^n)+(\boldsymbol{U}^{n+1}\cdot\bar{\boldsymbol{\xi}}^{n+1}+P_h\bar{\boldsymbol{u}}^{n+1}\cdot\boldsymbol{\xi}^{n+1}$$
$$+\boldsymbol{U}^n\cdot\bar{\boldsymbol{\xi}}^n+P_h\bar{\boldsymbol{u}}^n\cdot\boldsymbol{\xi}^n)(p_h^l u_l^{n+1}+p_h^l u_l^n)$$

and

$$(|\boldsymbol{u}^{n+1}|^2+|\boldsymbol{u}^n|^2)(u_l^{n+1}+u_l^n)-(|P_h\boldsymbol{u}^{n+1}|^2+|P_h\boldsymbol{u}^n|^2)(p_h^l u_l^{u+1}+p_h^l u_l^n)$$
$$=(|\boldsymbol{u}^{n+1}|^2+|\boldsymbol{u}^n|^2)(\eta_l^{n+1}+\eta_l^n)$$
$$+(\boldsymbol{u}^{n+1}\cdot\bar{\boldsymbol{\eta}}^{n+1}+P_h\bar{\boldsymbol{u}}^{n+1}\cdot\boldsymbol{\eta}^{n+1}+\boldsymbol{u}^n\cdot\bar{\boldsymbol{\eta}}^n+P_h\bar{\boldsymbol{u}}^n\cdot\boldsymbol{\eta}^n)(p_h^l u_l^{n+1}+p_h^l u_l^n),$$

we have

$$\Big|\frac{1}{4}(\beta(x)(|\boldsymbol{U}^{n+1}|^2+|\boldsymbol{U}^n|^2)(U_l^{n+1}+U_l^n),\ \xi_l^{n+1}+\xi_l^n)$$
$$-\frac{1}{4}(\beta(x)(|P_h\boldsymbol{u}^{n+1}|^2+|P_h\boldsymbol{u}^n|^2)(p_h^l u_l^{n+1}+p_h^l u_l^n),\ \xi_l^{n+1}+\xi_l^n)\Big|$$
$$<\frac{1}{2}B_0(\|\boldsymbol{U}^{n+1}\|_{L_\infty}^2+\|\boldsymbol{U}^n\|_{L_\infty}^2)(\|\xi_l^{n+1}\|_{L_2}^2+\|\xi_l^n\|_{L_2}^2)$$
$$+\frac{1}{4}B_0[(\|\boldsymbol{U}^{n+1}\|_{L_\infty}+\|P_h\boldsymbol{u}^{n+1}\|_{L_\infty})(\|\boldsymbol{\xi}^{n+1}\|_{L_2}^2+\|\xi_l^{n+1}\|_{L_2}^2+\|\xi_l^n\|_{L_2}^2)$$
$$+(\|\boldsymbol{U}^n\|_{L_\infty}+\|P_h\boldsymbol{u}^n\|_{L_\infty})(\|\boldsymbol{\xi}^n\|_{L_2}^2+\|\xi_l^{n+1}\|_{L_2}^2+\|\xi_l^n\|_{L_2}^2)](\|p_h^l u_l^{n+1}\|_{L_\infty}+\|p_h^l u_l^n\|_{L_\infty}),$$

$$\left|\frac{1}{4}(\beta(x)(|\boldsymbol{u}^{n+1}|^2+|\boldsymbol{u}^n|^2)(u_l^{n+1}+u_l^n),\ \xi_l^{n+1}+\xi_l^n)\right.$$
$$\left.-\frac{1}{4}(\beta(x)(|P_h\boldsymbol{u}^{n+1}|^2+|P_h\boldsymbol{u}^n|^2)(p_h^lu_l^{n+1}+p_h^lu_l^n),\ \xi_l^{n+1}+\xi_l^n)\right|$$
$$\leqslant\frac{1}{4}B_0(\|\boldsymbol{u}^{n+1}\|_{L_\infty}^2+\|\boldsymbol{u}^n\|_{L_\infty}^2)(\|\eta_l^{n+1}\|_{L_2}^2+\|\eta_l^n\|_{L_2}^2+\|\xi_l^{n+1}\|_{L_2}^2+\|\xi_l^n\|_{L_2}^2)$$
$$+\frac{1}{4}B_0(\|p_h^lu_l^{n+1}\|_{L_\infty}+\|p_h^lu_l^n\|_{L_\infty})\cdot[(\|\boldsymbol{u}^{n+1}\|_{L_\infty}+\|P_h\boldsymbol{u}^{n+1}\|_{L_\infty})(\|\boldsymbol{\eta}^{n+1}\|_{L_2}^2$$
$$+\|\xi_l^{n+1}\|_{L_2}^2+\|\xi_l^n\|_{L_2}^2)+(\|\boldsymbol{u}^n\|_{L_\infty}+\|P_h\boldsymbol{u}^n\|_{L_\infty})(\|\boldsymbol{\eta}^n\|_{L_2}^2+\|\xi_l^{n+1}\|_{L_2}^2+\|\xi_l^n\|_{L_2}^2)].$$

Moreover

$$\left|\frac{1}{2}\sum_{j=1}^N(K_{lj}(\cdot,\ t^{n+\frac{1}{2}})(\xi_j^{n+1}+\xi_j^n),\ \xi_l^{n+1}+\xi_l^n)\right|$$
$$\leqslant\frac{K}{2}\sum_{j=1}^N(\|\xi_j^{n+1}\|_{L_2}^2+\|\xi_j^n\|_{L_2}^2+\|\xi_l^{n+1}\|_{L_2}^2+\|\xi_l^n\|_{L_2}^2),$$
$$\left|\frac{1}{2}\sum_{j=1}^N(K_{lj}(\cdot,\ t^{n+\frac{1}{2}})(\eta_j^{n+1}+\eta_j^n),\ \xi_l^{n+1}+\xi_l^n)\right|$$
$$\leqslant\frac{K}{2}\sum_{j=1}^N(\|\eta_j^{n+1}\|_{L_2}^2+\|\eta_j^n\|_{L_2}^2+\|\xi_l^{n+1}\|_{L_2}^2+\|\xi_l^n\|_{L_2}^2),$$
$$\left|i\left(\frac{\eta_l^{n+1}-\eta_l^n}{k},\ \xi_l^{n+1}+\xi_l^n\right)\right|\leqslant\left\|\frac{\eta_l^{n+1}-\eta_l^n}{k}\right\|_{L_2}^2+\|\xi_l^{n+1}\|_{L_2}^2+\|\xi_l^n\|_{L_2}^2,$$
$$|(O(k^2),\ \xi_l^{n+1}+\xi_l^n)|\leqslant C(k^4+\|\xi_l^{n+1}\|_{L_2}^2+\|\xi_l^n\|_{L_2}^2).$$

Using the elliptic projection (2.2), we have

$$\frac{1}{2}A_l(\eta_l^{n+1}+\eta_l^n,\ \xi_l^{n+1}+\xi_l^n)+\frac{1}{2}(\alpha_l(\bar{\eta}_l^{n+1}+\bar{\eta}_l^n)_x,\ \xi_l^{n+1}+\xi_l^n)$$
$$=-\frac{1}{2}((K_{ll}^{n+1}+\widetilde{C})\eta_l^{n+1}+(K_{ll}^n+\widetilde{C})\eta_l^n,\ \xi_l^{n+1}+\xi_l^n).$$

Thus

$$\left|\frac{1}{2}A_l(\eta_l^{n+1}+\eta_l^n,\ \xi_l^{n+1}+\xi_l^n)+\frac{1}{2}(\alpha_l(\bar{\eta}_l^{n+1}+\bar{\eta}_l^n)_x,\ \xi_l^{n+1}+\xi_l^n)\right|$$
$$\leqslant\frac{1}{2}(K+\widetilde{C})(\|\eta_l^{n+1}\|_{L_2}^2+\|\eta_l^n\|_{L_2}^2+\|\xi_l^{n+1}\|_{L_2}^2+\|\xi_l^n\|_{L_2}^2).$$

Taking the imaginary part on both sides of equality (3.7) and substituting the above estimates into (3.7) and summing up for l from 1 to N, we have

$$\|\boldsymbol{\xi}^{n+1}\|_{L_2}^2-\|\boldsymbol{\xi}^n\|_{L_2}^2\leqslant Ck\left(\|\boldsymbol{\xi}^{n+1}\|_{L_2}^2+\|\boldsymbol{\xi}^n\|_{L_2}^2+\|\boldsymbol{\eta}^{n+1}\|_{L_2}^2+\|\boldsymbol{\eta}^n\|_{L_2}^2+\left\|\frac{\boldsymbol{\eta}^{n+1}-\boldsymbol{\eta}^n}{k}\right\|_{L_2}^2+k^4\right). \tag{3.8}$$

Here we have used Lemma 7, Corollary 2 and

$$\|P_h\boldsymbol{u}^n\|_{L_\infty}\leqslant\|P_h\boldsymbol{u}^n-\boldsymbol{u}^n\|_{L_\infty}+\|\boldsymbol{u}^n\|_{L_\infty}\leqslant C.$$

Now summing up for n from 1 to $H-1$ on both sides of inequality (3.8), we obtain

$$\|\boldsymbol{\xi}^H\|_{L_2}^2\leqslant Ak\sum_{j=0}^{H-1}\|\boldsymbol{\xi}^j\|_{L_2}^2+B\int_0^T\left(\|\boldsymbol{\eta}(t)\|_{L_2}^2+\left\|\frac{\partial\boldsymbol{\eta}(t)}{\partial t}\right\|_{L_2}^2\right)dt+Ck^4,$$

where A, B and C are constants independent of k. According to the Gronwall inequality, we have

$$\|\xi^H\|_{L_2}^2 \leqslant e^{AT}\left(B\int_0^T\left(\|\eta(t)\|_{L_2}^2+\left\|\frac{\partial\eta(t)}{\partial t}\right\|_{L_2}^2\right)dt+Ck^4\right), \quad 0\leqslant H\leqslant\left[\frac{T}{k}\right]. \tag{3.9}$$

From (3.9) and noticing

$$\|U^H-u(Hk)\|_{L_2}\leqslant\|\xi^H\|_{L_2}+\|\eta^H\|_{L_2} \text{ and } \left\|\frac{\partial\eta(t)}{\partial t}\right\|_{L_2}\leqslant Ch^r, \quad 0\leqslant t\leqslant T,$$

by Lemma 5, we obtain

Theorem 2. *Suppose* $q(s)=s$ *in* (1.2), *the conditions of Theorem* 1 *are satisfied, and* $u_{tt}\in L^\infty(0, T; \tilde{H}^1)$. *Then we have*

$$\|U^H-u(Hk)\|_{L_2}\leqslant C(k^2+h^r), \quad 0\leqslant H\leqslant\left[\frac{T}{k}\right].$$

For general $q(s)$, if $|q(s)|\leqslant As$, where A is a constant, $q(s)\geqslant 0$, $q'(s)\geqslant 0$, $s\in[0, \infty)$, the fully discrete scheme shall take the following form

$$\begin{aligned}
&i\left(\frac{U_l^{n+1}-U_l^n}{k}, \chi\right)+\frac{1}{2}A_l(U_l^{n+1}+U_l^n, \chi)+\frac{1}{2}(\alpha_l(\bar{U}_l^{n+1}+\bar{U}_l^n)_x, \chi)\\
&\quad+\frac{1}{2}(\beta(x)q(|U^{n+1}|^2)(U_l^{n+1}+U_l^n), \chi)+\frac{1}{2}\sum_{j=1}^N(K_{lj}(\cdot, t^{n+\frac{1}{2}})(U_j^{n+1}+U_j^n), \chi)\\
&=(G_l(\cdot, t^{n+\frac{1}{2}}), \chi), \quad \forall\chi\in U_h+iU_h, \ l=1, 2, \cdots, N.
\end{aligned} \tag{3.10}$$

Let $Q(s)=\int_0^s q(z)dz$. Suppose there exists $h_0>0$ so that $\int_I Q(|u_h^0|^2)dx<\infty$ for $0<h<h_0$. We can obtain the following estimate

$$\|U^H-u(Hk)\|_{L_2}\leqslant C(k+h^r), \quad 0\leqslant H\leqslant\left[\frac{T}{k}\right].$$

We omit its proof because it is similar to that of Theorem 2.

Finally we discuss the solvability of systems of fully discrete nonlinear equations.

For given $Y\in(U_h+iU_h)^N$ the continuous map $W=F(Y)$ which maps $(U_h+iU_h)^N$ onto itself is defined as

$$\begin{aligned}
&i\left(\frac{W_l-U_l^n}{k}, \chi\right)+\frac{1}{2}A_l(W_l+U_l^n, \chi)+\frac{1}{2}(\alpha_l(\bar{Y}_l+\bar{U}_l^n)_x, \chi)\\
&\quad+\frac{1}{2}(\beta(x)q(|Y|^2)(Y_l+U_l^n), \chi)-\frac{1}{2}(\beta(x)q(|Y|^2)(Y_l-W_l), \chi)\\
&\quad+\frac{1}{2}\sum_{j=1}^N(K_{lj}(\cdot, t^{n+\frac{1}{2}})(Y_j+U_j^n), \chi)\\
&=(G_l(\ , t^{n+\frac{1}{2}}), \chi), \quad \forall\chi\in U_h+iU_h, \ l=1, 2, \cdots, N.
\end{aligned}$$

Similarly to [12] we can prove that the map $W=F(Y)$ has a fixed point and it is $W=U^{n+1}$ Therefore we obtain the solvability of system (3.10).

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