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THE ITERATIVE ACCELERATIVE METHOD OF FINITE ELEMENT APPROXIMATION FOR THE SYSTEM $u=\sum u_j \frac{\partial u}{\partial x_j}+f$*

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Abstract

In this paper we use an iteration method[1] to get an approximate solution u^n and $\bar{u}^n$ which approximate the exact solution u with error estimates $\|u-u^n\|+ch\|u-u^n\|_1+\|u-\bar{u}^n\|_1\leqslant ch^{n+2}$.

Let us consider the following system:

$$\begin{cases} \Delta u=\sum_j u_j \frac{\partial u}{\partial x_j}+f, \\ u|_\Gamma=0 \end{cases} \tag{1}$$

in two dimension, Ω is a bounded domain with boundary Γ sufficiently smooth. The weak form of (1) is:

$$\begin{cases} u\in (H_0^1(\Omega))^2=V, \\ (u,\ v)_1+\left(\sum_j u_j \frac{\partial u}{\partial x_j}+f,\ v\right)=0, \quad \forall v\in V. \end{cases} \tag{1$'$}$$

We assume that u is an isolated solution, i.e. the linear problem

$$\begin{cases} w\in V, \\ (w,\ v)_1+\left(\sum_j w_j \frac{\partial u}{\partial x_j},\ v\right)+\left(\sum_j u_j \frac{\partial w}{\partial x_j},\ v\right)=0, \quad \forall v\in V \end{cases} \tag{2}$$

has only a trivial solution $w=0$ in V.

Let $u^0\in S_h\subset V$ be the finite element solution[2] of the corresponding Galerkin problem, i.e.

$$\begin{cases} u^0\in S_h\subset V, \\ (u^0,\ v)_1+\left(\sum_j u_j^0 \frac{\partial u^0}{\partial x_j}+f,\ v\right)=0, \quad \forall v\in S_h, \end{cases} \tag{3}$$

where S_h is finite element subspace with piecewise linear polynomial.

Let $\bar{u}^0$ be the solution of the following Poisson problem

$$\begin{cases} \bar{u}^0\in V, \\ (\bar{u}^0,\ v)_1+\left(\sum_j u_j^0 \frac{\partial u^0}{\partial x_j}+f,\ v\right)=0, \quad \forall v\in V \end{cases} \tag{4}$$

and $u^1=\bar{u}^0+\varphi^0$ with $\varphi^0\in S_h$ and

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$$(\varphi^0, v)_1+\Big(\sum_j \frac{\partial u^0}{\partial x_j}(\varphi_j^0+\bar{u}_j^0-u_j^0)+\sum_j u_j^0\frac{\partial}{\partial x_j}(\varphi^0+\bar{u}^0-u^0), v\Big)=0, \quad \forall v\in S_h. \tag{5}$$

Problem (5) has a unique solution which will be proved below.

Let

$$u^{n+1}=\bar{u}^n+\varphi^n, \quad \bar{u}^n\in V \text{ such that} \tag{6}$$

$$(\bar{u}^n, v)_1+\Big(\sum_j u_j^n\frac{\partial u^n}{\partial x_j}+f, v\Big)=0, \quad \forall v\in V \tag{7}$$

and $\varphi^n\in S_h$ such that

$$(\varphi^n, v)_1+\Big(\sum_j \frac{\partial u^0}{\partial x_j}(\varphi_j^n+\bar{u}_j^n-u_j^n)+\sum_j u_j^0\frac{\partial}{\partial x_j}(\varphi^n+\bar{u}^n-u^n), v\Big)=0, \quad \forall v\in S_h. \tag{8}$$

Let us define the operator K, $w=Kg$ by

$$\begin{cases} w\in V, \\ (w, v)_1=(g, v), \quad \forall v\in V, g\in (L^2(\Omega))^2. \end{cases}$$

It is easy to see that K: $(L^2(\Omega))^2\to(H^2(\Omega)\cap H_0^1(\Omega))^2$ and

$$K:\ [(H_0^1(\Omega))^2]'=V'\to V.$$

Set

$$L\varphi=\sum_j \frac{\partial u^0}{\partial x_j}\varphi_j+\sum_j u_j^0\frac{\partial \varphi}{\partial x_j}.$$

So

$$L:\ (H^1(\Omega))^2\to(L^2(\Omega))^2$$

is linear continuous operator, we will prove in Lemma 3 that L: $(L^2(\Omega))^2\to[H_0^1(\Omega))^2]'=V'$ is linear continuous operator, i.e.,

$$\|L\varphi\|_{V'}=\sup_{v\in V}\frac{|\langle L\varphi, v\rangle|}{\|v\|_1}\le C\|\varphi\|. \tag{9}$$

Problem (8) can be rewritten in operator form

$$\varphi^n+pKL\varphi^n+pKL(\bar{u}^n-u^n)=0, \tag{10}$$

where p is an orthogonal projection onto subspace S_h with the scalar product $(\ ,\ )_1$.

Problem (2) can be rewritten in operator form

$$w+K\tilde{L}w=0, \tag{11}$$

where

$$\tilde{L}w+\sum_j\Big(\frac{\partial u}{\partial x_j}w_j+u_j\frac{\partial w}{\partial x_j}\Big). \tag{12}$$

As u is an isolated solution, $I+K\tilde{L}$ has bounded inverse $(I+K\tilde{L})^{-1}$ in $(H^1(\Omega))^2$. Since $u^0\to u$[2], we can prove that $(I+KL)$ has a bounded inverse operator $(I+KL)^{-1}$ and the norm $\|(I+KL)^{-1}\|$ is uniformly bounded. This conclusion will be proved in Lemma 2. By using the operator K and the projection operator p, the problem (1)′, (3), (4), (7) can be rewritten in operator form

$$u+K\Big(\sum_j u_j\frac{\partial u}{\partial x_j}+f\Big)=0, \tag{13}$$

$$u^0+pK\Big(\sum_j u_j^0\frac{\partial u^0}{\partial x_j}+f\Big)=0, \tag{14}$$

$$\bar{u}^0+K\Big(\sum_j u_j^0\frac{\partial u^0}{\partial x_j}+f\Big)=0, \tag{15}$$

$$\bar{u}^n+K\Big(\sum_j u_j^n\frac{\partial u^n}{\partial x_j}+f\Big)=0, \tag{16}$$

$$\begin{aligned}(I+pKL)u^{n+1}&=(I+pKL)(\bar{u}^n+\varphi^n)=(I+pKL)\varphi^n+\bar{u}^n+pKL\bar{u}^n\\&=-pKL(\bar{u}^n-u^n)+pKL\bar{u}^n-K\Big(\sum_j u_j^n\frac{\partial u^n}{\partial x_j}+f\Big)\\&=pKLu^n-K\Big(\sum_j u_j^n\frac{\partial u^n}{\partial x_j}+f\Big),\end{aligned}$$

$$(I+pKL)u=-K\Big(\sum_j u_j\frac{\partial u}{\partial x_j}+f\Big)+pKLu,$$

$$(I+pKL)(u^{n+1}-u)=-K\Big(\sum_j u_j^n\frac{\partial u^n}{\partial x_j}-u_j\frac{\partial u}{\partial x_j}\Big)+pKL(u^n-u), \tag{17}$$

$$\begin{aligned}(I+pKL)(u^{n+1}-u)&=-(I-p)KL(u^n-u)+KL(u^n-u)\\&\quad-K\Big(\sum_j u_j^n\frac{\partial u^n}{\partial x_j}-u_j\frac{\partial u}{\partial x_j}\Big)\\&=-(I-p)KL(u^n-u)\\&\quad+K\Big(\sum_j\frac{\partial u^0}{\partial x_j}(u_j^n-u_j)+\sum_j u_j^0\frac{\partial}{\partial x_j}(u^n-u)\Big)\\&\quad-K\Big(\sum_j\frac{\partial u^n}{\partial x_j}(u_j^n-u_j)+\sum_j u_j\frac{\partial}{\partial x_j}(u^n-u)\Big).\end{aligned} \tag{18}$$

As

$$I+pKL=I+KL-(I-p)KL, \tag{19}$$

$$\|(I-p)KLv\|_1\leqslant ch\|KLv\|_2\leqslant ch\|Lv\|\leqslant ch\|v\|_1.$$

So $\|(I-p)KL\|_{1\to1}\leqslant ch$ and $I+pKL$ has a bounded inverse operator; the norm $\|(I+pKL)^{-1}\|$ is uniformly bounded. Hence problem (8) has a unique solution φ^n.

Note that

$$(I+pKL)(u^{n+1}-u)=(I+KL)(u^{n+1}-u)-(I-p)KL(u^{n+1}-u),$$

$$\|(I-p)KL(u^{n+1}-u)\|\leqslant ch^2\|KL(u^{n+1}-u)\|_2\leqslant ch^2\|u^{n+1}-u\|_1. \tag{20}$$

By (19) we can rewrite (18) as

$$\begin{aligned}&(I+KL)(u^{n+1}-u)\\&=-(I-p)KL(u^n-u)+K\Big(\sum_j\frac{\partial u^0}{\partial x_j}(u_j^n-u_j)+\sum_j u_j^0\frac{\partial}{\partial x_j}(u^n-u)\Big)\\&\quad+(I-p)KL(u^{n+1}-u)-K\Big(\sum_j\frac{\partial u^n}{\partial x_j}(u_j^n-u_j)+\sum_j u_j\frac{\partial}{\partial x_j}(u^n-u)\Big).\end{aligned} \tag{21}$$

Using (20), (21) and $\|Kfg\|\leqslant c\|f\|\|g\|$[1] we have

$$\begin{aligned}\|u^{n+1}-u\|&\leqslant ch^2(\|u^{n+1}-u\|_1+\|u^n-u\|_1)+c\|u^n-u\|(\|u^n-u\|_1+\|u-u^0\|_1)\\&\quad+c\|u^n-u\|_1\|u-u^0\|\\&\leqslant ch^2(\|u^{n+1}-u\|_1+\|u^n-u\|_1)+ch\|u^n-u\|+c\|u^n-u\|\,\|u^n-u\|_1,\end{aligned} \tag{22}$$

where we have used

$$\|u-u^0\|+ch\|u-u^0\|_1\leqslant ch^2\ \text{[2]}. \tag{23}$$

By using (17) and $\|(I+pKL)^{-1}\|\leqslant c$ (uniformly) we have

$$\|u^{n+1}-u\|_1\leqslant c\Big\|K\Big(\sum_j u_j^n\frac{\partial u^n}{\partial x_j}-u_j\frac{\partial u}{\partial x_j}\Big)\Big\|_1+c\|KL(u^n-u)\|_1. \tag{24}$$

Let

$$w=K\Big(\sum_j u_j^n \frac{\partial u^n}{\partial x_j}-u_j\frac{\partial u}{\partial x_j}\Big)$$

i.e.

$$\begin{cases} w\in (H_0^1(\Omega))^2=V, \\ (w,\ v)_1=\Big(\sum_j u_j^n\frac{\partial u^n}{\partial x_j}-u_j\frac{\partial u}{\partial x_j},\ v\Big),\quad \forall v\in V, \end{cases} \tag{25}$$

$$(w,\ v)_1=\Big(\sum_j (u_j^n-u_j)\frac{\partial u^n}{\partial x_j}+\sum_j u_j\frac{\partial}{\partial x_j}(u^n-u),\ v\Big)$$

$$=\Big(\sum_j (u_j^n-u_j)\frac{\partial u^n}{\partial x_j},\ v\Big)-\Big(\sum_j u_j(u^n-u),\ \frac{\partial v}{\partial x_j}\Big)-\Big(\sum_j\frac{\partial u_j}{\partial x_j}(u^n-u),\ v\Big),$$

$$\|w\|_1^2\leqslant c\|u^n-u\|(\|u^n\|_{1,4}+\|u\|_{0,\infty}+\|u\|_{1,4})\|w\|_1$$

$$\leqslant c\|u^n-u\|(\|u^n\|_{1,4}+\|u\|_2)\|w\|_1,$$

$$\|w\|_1\leqslant c\|u^n-u\|$$

i.e.

$$\Big\|K\Big(\sum_j u_j^n\frac{\partial u^n}{\partial x_j}-u_j\frac{\partial u}{\partial x_j}\Big)\Big\|_1\leqslant c\|u^n-u\|, \tag{26}$$

where we have used $\|u^n\|_{1,4}\leqslant c$ (uniformly), which will be proved in Lemma 1.

Using the same method as in proving (26) we can prove

$$\|KL(u^n-u)\|_1\leqslant c\|u^n-u\|. \tag{27}$$

By (24), (26), (27) we get

$$\|u^{n+1}-u\|_1\leqslant c\|u^n-u\|. \tag{28}$$

From (22), (23), (28) we derive the conclusion

$$\|u^n-u\|+ch\|u^n-u\|_1\leqslant ch^{n+2}.$$

Using the same method as in proving (26) we have

$$\|\bar{u}^n-u\|_1=\Big\|K\Big(\sum_j u_j^n\frac{\partial u^n}{\partial x_j}-u_j\frac{\partial u}{\partial x_j}\Big)\Big\|_1\leqslant c\|u-u^n\|\leqslant ch^{n+2}.$$

Therefore we have the following theorem.

Theorem. *Assume the u is an isolated solution of problem (1)′, and $(u^n, \bar{u}^n)$ is defined as above. Then*

$$\|u-u^n\|+\|u-\bar{u}^n\|_1+ch\|u-u^n\|_1\leqslant ch^{n+2},$$

where c is independent on h.

Lemma 1. *Under the condition of the theorem we have*

$$\|u^n\|_{0,\infty}\leqslant c,\quad \|u^n\|_{1,4}\leqslant c^{[8]},$$

where c is independent on h.

Lemma 2. *Assume that problem (2) has a unique solution. Then the problem $(I+KL)\varphi=0$ has only a trivial solution $\varphi=0$.*

Proof. As problem (2) is equivalent to

$$w+K\tilde{L}w=0 \tag{11}$$

by the definition of the operators L and $\tilde{L}$ we have

$$K(L-\tilde{L})v=K\Big(\sum_j u_j^0\frac{\partial v}{\partial x_j}+v_j\frac{\partial u^0}{\partial x_j}\Big)-K\Big(\sum_j u_j\frac{\partial v}{\partial x_j}+v_j\frac{\partial u}{\partial x_j}\Big)$$

$$=K\Big[\sum_j (u_j^0-u_j)\frac{\partial v}{\partial x_j}+v_j\frac{\partial}{\partial x_j}(u^0-u)\Big].$$

Using the same method as in proving (26), we can prove

$$\|K(L-\tilde{L})v\|_1 \leqslant c\|u^0-u\|_1\|v\|_1 \leqslant ch\|v\|_1.$$

So $I+KL$ has a bounded inverse operator and the norm $\|(I+KL)^{-1}\|$ is uniformly bounded.

Lemma 3. $\|L\varphi\|_{((H_0^1(\Omega))^2)'} \leqslant c\|\varphi\|, \quad \forall\varphi \in (L^2(\Omega))^2.$

Proof.

$$\langle L\varphi, v\rangle = \left\langle \sum_j \left(\varphi_j \frac{\partial u^0}{\partial x_j} + u_j^0 \frac{\partial \varphi}{\partial x_j}, v\right)\right\rangle$$

$$= \int_\Omega \sum_j \varphi_j \frac{\partial u^0}{\partial x_j} v\,dx - \int_\Omega \sum_j \frac{\partial u^0}{\partial x_j} \varphi v\,dx - \int_\Omega \sum_j u_j^0 \varphi \frac{\partial v}{\partial x_j}\,dx, \quad \forall v \in V.$$

So

$$|\langle L\varphi, v\rangle| \leqslant c\|\varphi\|(\|u^0\|_{1,4}\|v\|_{0,4} + \|u^0\|_{0,\infty}\|v\|_1) \leqslant c\|\varphi\|\|v\|_1.$$

Therefore

$$\|L\varphi\|_{V'} \leqslant c\|\varphi\|, \quad \forall\varphi \in (L^2(\Omega))^2.$$

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