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A POSTERIORI ERROR ESTIMATE OF FINITE ELEMENT METHOD FOR THE OPTIMAL CONTROL WITH THE STATIONARY BÉNARD PROBLEM*

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Abstract

In this paper, we consider the adaptive finite element approximation for the distributed optimal control associated with the stationary Bénard problem under the pointwise control constraint. The states and co-states are approximated by polynomial functions of lowest-order mixed finite element space or piecewise linear functions and control is approximated by piecewise constant functions. We give the a posteriori error estimates for the control, the states and co-states.

Mathematics subject classification: 49J20, 65N30.
Key words: Optimal control problem, Stationary Bénard problem, Nonlinear coupled system, A posteriori error estimate.

1. Introduction

The control of viscous flow for the purpose of achieving some desired objective is crucial to many technological and scientific applications. The Boussinesq approximation of the Navier-Stokes system is frequently used as mathematical model for fluid flow in semiconductor melts. In many crystal growth technics, such as Czochralski growth and zone-melting technics, the behavior of the flow has considerable impact on the crystal quality. It is therefore quite natural to establish flow conditions that guarantee desired crystal properties. As control actions, they include distributed forcing, distributed heating, and others. For example, the control of vorticity has significant applications in science and engineering such as the control of turbulence and control of crystal growth process.

Considerable progress has been made in mathematics physics and computation for the optimal control problems of the viscous flow; see [1, 2, 9, 11, 12] and reference therein. Optimal control problems of the thermally coupled incompressible Navier-Stokes equation by Neumann and Dirichlet boundary heat controls were considered in [11, 12]. Also, the time dependent problems were considered in the literature. In this article, we consider the Bénard problem whose state is governed by the Boussinesq equations, which are crucial to many technological and scientific applications. Without the control constraint, the approximation for the optimal control of the stationary Bénard problem was considered in [16], and it used the gradient iterative method to solve the discretized equations. For the constrained control case, there seems to

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be little work on this problem. This paper is concerned with the finite element approximation for the constrained optimal control problem of the stationary Bénard problem:

$$(\mathcal{P}) \quad \min_{Q\in K} \quad J(Q) = \{\frac{1}{2}\|\mathbf{u}-\mathbf{U}\|^2_{\mathbf{L}^2} + \frac{\alpha}{2}\|Q\|^2_{0,\Omega}\},$$

subject to the Boussinesq system:

$$\begin{aligned}
&(a) \quad -\nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = T\mathbf{g} + f \quad in\ \Omega,\\
&(b) \quad \nabla\cdot\mathbf{u} = 0 \quad in\ \Omega,\\
&(b) \quad -\kappa\Delta T + \mathbf{u}\cdot\nabla T = Q \quad in\ \Omega,\\
&(c) \quad \mathbf{u} = 0 \quad T = 0 \quad on\ \partial\Omega
\end{aligned} \tag{1.1}$$

and subject to the control constraint

$$K = \Big\{Q \in L^2(\Omega) : Q(x) \geq d > 0;\ a.e.\ x \in \Omega\Big\}, \tag{1.2}$$

where Ω is the regular bounded and convex open set in $\mathbb{R}^n$ ($n = 2,\ or\ 3$), with $\partial\Omega \in C^{1,1}$. $\mathbf{u}, p, T$ denote the velocity, pressure and temperature fields, respectively, f is a body force, and the control Q. The vector $\mathbf{g}$ is in the direction of gravitational acceleration and $\kappa > 0$ is the thermal conductivity parameter. In this paper we only consider, for the simplicity, the case where κ is constant. Assume $\nu > 0$ is the kinematic viscosity.

The optimal control problem $(\mathcal{P})$ is to seek the state variables $(\mathbf{u}, p, T)$ and Q such that the functional J is minimized subject to (1.1) where $\mathbf{U}$ is some desired velocity fields. The physical target of the minimization problem is to match a desired flow field by adjusting the distributed control Q.

Adaptive finite element approximation is of very importance in improving accuracy and efficiency of the finite element discretisation. It ensures a higher density of nodes in certain area of the given domain, where the solution is more difficult to approximate, using a posteriori error indicator. In this sense, efficiency and reliability of adaptive finite element approximation rely very much on the error indicator used. Recently adaptive mesh refinement has been found quite useful in computing optimal control problem governed by elliptic equations, see [19], for example. Usually the control variable has only limited regularity. Thus suitable adaptive mesh can quite efficiently reduce the approximation error. There have been very extensive studies on the a posteriori error estimates and convergence analysis for the optimal control problems governed by elliptic or time dependent equations; see, for example, [22,24,25] and [19,23,26] and the references cited therein. However there seems to exist few known results on the a posteriori error estimates for the above control problem governed by the coupled nonlinear equations.

The paper is organized as follows. In Section 2, we give some notations and assumptions that will be used throughout the paper. In Section 3, we will discuss the finite element approximation for the optimal control problem. Section 4 contains the a posteriori error estimate for the optimal control problem in this article.

2. Notations and Preliminaries

Using the classical techniques, it can be proved that the optimal control problem has at least one solution. The reader is referred to [15, 18] for the details.

Similarly to [17, 19] and using the result of [8], it is well known that if $(\mathbf{u}, p, T)$ is the solution of $(\mathcal{P})$, then there are the co-state $(\mathbf{w}, \sigma, \varphi, Q)$ such that $(\mathbf{u}, p, T, \mathbf{w}, \sigma, \varphi, Q)$ satisfies the following optimality conditions:

$$
\begin{aligned}
&(a) \quad -\nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = T\mathbf{g} + f \quad in\ \Omega,\\
&(b) \quad \nabla\cdot\mathbf{u} = 0 \quad in\ \Omega,\\
&(b) \quad -\kappa\Delta T + \mathbf{u}\cdot\nabla T = Q \quad in\ \Omega,\\
&(c) \quad \mathbf{u} = 0 \quad T = 0 \quad on\ \partial\Omega
\end{aligned} \tag{2.1}
$$

coupled with the co-state equations and variational inequality:

$$
\begin{aligned}
&(a) \quad -\nu\Delta\mathbf{w} - (\mathbf{u}\cdot\nabla)\mathbf{w} + \nabla\mathbf{u}^{tr}\mathbf{w} - \nabla\sigma + \varphi\nabla T = \mathbf{u} - \mathbf{U} \quad in\ \Omega,\\
&(b) \quad \nabla\cdot\mathbf{w} = 0 \quad in\ \Omega,\\
&(c) \quad -\kappa\Delta\varphi - \mathbf{u}\cdot\nabla\varphi = \mathbf{w}\cdot\mathbf{g} \quad in\ \Omega,\\
&(c) \quad \mathbf{w} = 0 \quad \varphi = 0 \quad on\ \partial\Omega,\\
&(d) \quad \int_\Omega (\alpha Q + \varphi)(P - Q)\,dx \geq 0 \quad \forall\ P \in K,
\end{aligned} \tag{2.2}
$$

where $\nabla\mathbf{u}^{tr}$ denotes the transpose of $\nabla\mathbf{u}$.

To consider the weak formulations of the equations (2.1) and (2.2), we need to introduce some function spaces and the bilinear and trilinear forms. In this paper we adopt the standard notation $W^{m,q}(\Omega)$ for Sobolev spaces on Ω with the norm $\|\cdot\|_{m,q,\Omega}$ and the seminorm $|\cdot|_{m,q,\Omega}$. We denote $W^{m,2}(\Omega)$ $(W_0^{m,2}(\Omega))$ by $H^m(\Omega)$ $(H_0^m(\Omega))$ with the norm $\|\cdot\|_{m,\Omega}$ and the semi-norm $|\cdot|_{m,\Omega}$. For vector-valued functions and spaces of vector-valued functions, which are indicated by boldface, we define the Sobolev Space $\mathbf{H}^m(\Omega)$

$$
\mathbf{H}^m(\Omega) = \Big\{\mathbf{u} = (u_1, \cdots, u_n)|\ u_i \in H^m(\Omega),\ i = 1, \cdots, n\Big\}
$$

and its associated norm $\|\cdot\|_{\mathbf{H}^m(\Omega)}$ is given by

$$
\|\mathbf{u}\|^2_{\mathbf{H}^m(\Omega)} = \sum_{i=1}^n \|u_i\|^2_{H^m(\Omega)}.
$$

We also define the following subspaces

$$
L_0^2(\Omega) = \{f \in L^2(\Omega) :\ \int_\Omega f\,dx = 0\}, \quad \mathbf{H}_0^1(\Omega) = \{\mathbf{u} \in \mathbf{H}^1(\Omega);\ \mathbf{u} = 0\ \ on\ \partial\Omega\}.
$$

Then introduce the bilinear and trilinear forms, for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{H}^1(\Omega)$, $T, S \in H^1(\Omega)$ and $q \in L_0^2(\Omega)$,

$$
\begin{aligned}
a_0(\mathbf{u}, \mathbf{v}) &= \int_\Omega \nu\nabla\mathbf{u}\cdot\nabla\mathbf{v}\,dx, \quad a_1(T, S) = \int_\Omega \kappa\nabla T\cdot\nabla S\,dx,\\
c_0(\mathbf{u}, \mathbf{v}, \mathbf{w}) &= \int_\Omega (\mathbf{u}\cdot\nabla)\mathbf{v}\cdot\mathbf{w}\,dx, \quad c_1(\mathbf{u}, T, S) = \int_\Omega \mathbf{u}\cdot\nabla TS\,dx,
\end{aligned}
$$

and

$$
b(\mathbf{v}, q) = -\int_\Omega q\nabla\cdot\mathbf{v}\,dx, \quad d(T, \mathbf{v}) = \int_\Omega T\mathbf{g}\cdot\mathbf{v}\,dx.
$$

Moreover we assume that $b(\mathbf{v}, q)$ satisfies the *inf-sup* condition, i.e.: there exists a constant $\beta > 0$ such that

$$\inf_{0\neq q\in L_0^2(\Omega)} \sup_{\mathbf{0}\neq\mathbf{v}\in\mathbf{H}_0^1(\Omega)} \frac{b(\mathbf{v},q)}{\|\mathbf{v}\|_{\mathbf{H}^1}\|q\|_{L^2}} \geq \beta. \tag{2.3}$$

Then, we have the weak formulation: seek $(\mathbf{u}, p, T, \mathbf{w}, \sigma, \varphi, Q) \in \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega) \times H_0^1(\Omega) \times \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega) \times H_0^1(\Omega) \times K$ such that

$$\begin{aligned}
&(a)\quad a_0(\mathbf{u},\mathbf{v}) + c_0(\mathbf{u},\mathbf{u},\mathbf{v}) + b(\mathbf{v},p) = d(T,\mathbf{v}) + (f,\mathbf{v}) \quad \forall\ \mathbf{v}\in\mathbf{H}_0^1(\Omega),\\
&(b)\quad b(\mathbf{u},q) = 0 \quad \forall\ \ q\in L_0^2(\Omega),\\
&(c)\quad a_1(T,S) + c_1(\mathbf{u},T,S) = (Q,S) \quad \forall\ S\in H_0^1(\Omega)
\end{aligned} \tag{2.4}$$

and

$$\begin{aligned}
&(a)\quad a_0(\mathbf{w},\mathbf{v}) + c_0(\mathbf{v},\mathbf{u},\mathbf{w}) + c_0(\mathbf{u},\mathbf{v},\mathbf{w}) - b(\mathbf{v},\sigma)\\
&\qquad\qquad = (\mathbf{u}\text{-}\mathbf{U},\mathbf{v}) - c_1(\mathbf{v},T,\varphi) \quad \forall\ \mathbf{v}\in\mathbf{H}_0^1(\Omega),\\
&(b)\quad b(\mathbf{w},q) = 0 \quad \forall\ q\in L_0^2(\Omega),\\
&(c)\quad a_1(\varphi,S) + c_1(\mathbf{u},S,\varphi) = d(S,\mathbf{w}) \quad \forall\ S\in H_0^1(\Omega),\\
&(d)\quad \int_\Omega (\alpha Q + \varphi)(P - Q)\,dx \geq 0 \quad \forall\ P\in K.
\end{aligned} \tag{2.5}$$

3. Finite Element Approximation

We are now able to introduce a finite element approximation for the optimal control problem (1.1). The same as the article [3], we first give the following basic knowledge of finite element method. To this end, we consider a family of triangulations $\mathcal{T}_h$, $h > 0$, of $\bar{\Omega}$. With each element $\mathcal{T} \in \mathcal{T}_h$, we associate two parameters $\rho(\mathcal{T})$ and $\sigma(\mathcal{T})$, where $\rho(\mathcal{T})$ denotes the diameter of the set $\mathcal{T}$ and $\sigma(\mathcal{T})$ is the diameter of the largest ball contained in $\mathcal{T}$. The mesh size of the grid is defined by $h = \max_{\mathcal{T}\in\mathcal{T}_h} \rho(\mathcal{T})$. We suppose that triangulations $\mathcal{T}_h$ satisfy the following regularity assumptions:

(H_1) *There exist two positive constants ρ and σ such that*

$$\frac{\rho(\mathcal{T})}{\sigma(\mathcal{T})} \leq \sigma, \qquad \frac{\sigma(\mathcal{T})}{\rho(\mathcal{T})} \leq \rho$$

hold for all $\mathcal{T} \in \mathcal{T}_h$ and all $0 < h \leq 1$.

(H_2) *Define $\bar{\Omega}_h = \bigcup_{\mathcal{T}\in\mathcal{T}_h} \mathcal{T}$, and let Ω_h and Γ_h denote its interior and its boundary, respectively. We assume that $\bar{\Omega}_h$ is convex and that the vertices of $\mathcal{T}_h$ placed on the boundary of Γ_h are points of Γ. We also assume that*

$$|\Omega\backslash\Omega_h| \leq Ch^2.$$

Next, to every boundary triangle $\mathcal{T}$ of $\mathcal{T}_h$ we associate another triangle $\hat{\mathcal{T}}$ with curved boundary, in which the edge between boundary nodes of $\mathcal{T}$ is substituted by the corresponding curved part of Γ. We denote by $\hat{\mathcal{T}}_h$ the union of these curved boundary triangles with interior triangles of $\mathcal{T}_h$, such that $\bar{\Omega} = \bigcup_{\hat{\mathcal{T}}\in\hat{\mathcal{T}}_h} \hat{\mathcal{T}}$.

Denote by P_k function space of polynomial of degree less or equal than k. Introduce finite element spaces as follows:

$$K'_h = \Big\{Q_h \in L^2(\Omega) : Q_h|_{\hat{\mathcal{T}}} = constant, \quad \hat{\mathcal{T}} \in \hat{\mathcal{T}}_h\Big\}, \quad K_h = K'_h \cap K,$$
$$V_h = \Big\{y_h \in C(\bar{\Omega}) : y_h|_{\mathcal{T}} \in P_1(\mathcal{T}), \quad \mathcal{T} \in \mathcal{T}_h; \quad y_h = 0 \ \ on \ \ \bar{\Omega}\backslash\Omega_h\Big\}.$$

Next we introduce the one-order Raviart-Thomas mixed finite element spaces as [20]: $\bar{\mathbf{V}}_h \times \bar{X}_h \subset \mathbf{H}_0^1 \times L_0^2$ such that for a positive constant β_0, the following *inf-sup* condition satisfies:

$$\inf_{0\neq q_h \in \bar{X}_h} \sup_{\mathbf{0}\neq \mathbf{v}_h \in \mathbf{V}_h} \frac{b(\mathbf{v}_h, q_h)}{\|\mathbf{v}_h\|_{\mathbf{H}^1}\|q_h\|_{L^2}} \geq \beta_0. \tag{3.1}$$

Moreover, similarly to V_h, we define

$$\mathbf{V}_h = \Big\{\mathbf{y}_h \in \bar{\mathbf{V}}_h \quad on \ \Omega_h; \quad \mathbf{y}_h = 0 \ \ on \ \ \bar{\Omega}\backslash\Omega_h\Big\},$$
$$X_h = \Big\{p_h \in \bar{X}_h : p_h|_{\hat{\mathcal{T}}} = constant, \quad \hat{\mathcal{T}} \in \hat{\mathcal{T}}_h\Big\}.$$

Now, it is obvious that $\mathbf{V}_h \times X_h$ is defined on $\bar{\Omega}$, and then the finite dimensional approximation of the optimal control problem is:

$$(\mathcal{P}_h) \quad \min_{Q_h \in K_h} \quad J_h(Q_h) = \left\{\frac{1}{2}\|\mathbf{u}_h - \mathbf{U}\|^2_{\mathbf{L}^2} + \frac{\alpha}{2}\|Q_h\|^2_{0,\Omega}\right\} \tag{3.2}$$

subject to seek $(\mathbf{u}_h, p_h, T_h) \in \mathbf{V}_h \times X_h \times V_h$ such that

$$\begin{aligned}
&(a) \quad a_0(\mathbf{u}_h, \mathbf{v}_h) + c_0(\mathbf{u}_h, \mathbf{u}_h, \mathbf{v}_h) + b(\mathbf{v}_h, p_h) = d(T_h, \mathbf{v}_h) + (f, \mathbf{v}_h) \quad \forall\ \mathbf{v}_h \in \mathbf{V}_h,\\
&(b) \quad b(\mathbf{u}_h, q_h) = 0 \quad \forall\ q_h \in X^h,\\
&(c) \quad a_1(T_h, S_h) + c_1(\mathbf{u}_h, T_h, S_h) = (Q_h, S_h) \quad \forall\ S_h \in V_h.
\end{aligned} \tag{3.3}$$

The optimal control problem $(\mathcal{P}_h)$ associated with state equations (3.3) is equivalent to optimality conditions as follows: *Seek* $(\mathbf{u}_h, p_h, T_h) \in \boldsymbol{V}_h \times X_h \times V_h$ *such that*

$$\begin{aligned}
&(a) \quad a_0(\boldsymbol{u}_h, \boldsymbol{v}_h) + c_0(\boldsymbol{u}_h, \boldsymbol{u}_h, \boldsymbol{v}_h) + b(\boldsymbol{v}_h, p_h) = d(T_h, \boldsymbol{v}_h) + (f, \boldsymbol{v}_h) \quad \forall\ \boldsymbol{v}_h \in \boldsymbol{V}_h,\\
&(b) \quad b(\boldsymbol{u}_h, q_h) = 0 \quad \forall\ q_h \in X_h,\\
&(c) \quad a_1(T_h, S_h) + c_1(\boldsymbol{u}_h, T_h, S_h) = (Q_h, S_h) \quad \forall\ S_h \in V_h
\end{aligned} \tag{3.4}$$

couple with co-state system and inequality: $(\mathbf{w}_h, \sigma_h, \varphi_h, Q_h) \in \boldsymbol{V}_h \times X_h \times V_h \times K_h$ *such that*

$$\begin{aligned}
&(a) \quad a_0(\boldsymbol{w}_h, \boldsymbol{v}_h) + c_0(\boldsymbol{v}_h, \boldsymbol{u}_h, \boldsymbol{w}_h) + c_0(\boldsymbol{u}_h, \boldsymbol{v}_h, \boldsymbol{w}_h) - b(\boldsymbol{v}_h, \sigma_h)\\
&\qquad = (\boldsymbol{u}_h - \boldsymbol{U}, \boldsymbol{v}_h) - c_1(\boldsymbol{v}_h, T_h, \varphi_h) \quad \forall\ \boldsymbol{v}_h \in \boldsymbol{V}_h,\\
&(b) \quad b(\boldsymbol{w}_h, q_h) = 0 \quad \forall\ q_h \in X_h,\\
&(c) \quad a_1(\varphi_h, S_h) + c_1(\boldsymbol{u}_h, S_h, \varphi_h) = d(S_h, \boldsymbol{w}_h) \quad \forall\ S_h \in V_h,\\
&(d) \quad \int_\Omega (\alpha Q_h + \varphi_h)(P_h - Q_h)dx \geq 0 \quad \forall\ P_h \in K_h.
\end{aligned} \tag{3.5}$$

Based on the results of [15, 16] about the existing of solution, we know that there exists one solution of $(\mathcal{P})$ and $(\mathcal{P}_h)$ respectively at least. On the other hand, we denote constants C and ϵ be a generic constant and small positive number which are independent of the discrete parameters and may have different values in different circumstances respectively.

4. A Posteriori Error Estimates

In this section, we will give the a posteriori error estimates of control and states. Before that, we need to give some useful assumptions or results.

Firstly, we need assume that the cost function J is strictly convex near the solutions Q, i.e.,

(H_3): For each solution Q there is a neighborhood of Q in L^2 such that J is convex in the sense that there is a constant $c_* > 0$ satisfying:

$$c_* \|Q - P\|_{0,\Omega}^2 \leq (J'(Q) - J'(P), Q - P), \tag{4.1}$$

for all P in this neighborhood of Q.

The convexity of the cost function J is closely related to the second order sufficient conditions of the control problem, which are assumed in many studies on numerical methods of the problem, for example [3, 5, 8, 14]. More discussion of this can be found in, for example, [6] and [7].

Secondly, we introduce the definition of [25] that the solution $(Q, \mathbf{u}, T)$ is regular: which means that for the solution $\mathbf{u}$ of $(\mathcal{P})$, the linear co-state system

$$\begin{aligned}
&(a) \quad -\nu\Delta\mathbf{v} - (\mathbf{u}\cdot\nabla)\mathbf{v} + \nabla\mathbf{v}^{tr}\mathbf{u} - \nabla\zeta + \varrho\nabla T = \mathbf{r} \quad in\ \Omega, \\
&(b) \ \ \nabla\cdot\mathbf{v} = m \quad in\ \Omega, \\
&(c) \quad -\kappa\Delta\varrho - \mathbf{u}\cdot\nabla\varrho - \mathbf{v}\cdot\mathbf{g} = g \quad in\ \Omega, \\
&(d) \ \ \mathbf{v} = 0, \quad \varrho = 0 \quad on\ \partial\Omega
\end{aligned} \tag{4.2}$$

is well-posed and:

(R_1) For each $(\mathbf{r}, m, g) \in [\mathbf{H}^{-1}(\Omega)]^n \times L^2(\Omega) \times H^{-1}(\Omega)$, the system (4.2) has a unique solution and there holds the a priori estimate

$$\|\mathbf{v}\|_{\mathbf{H}^1} + \|\varrho\|_{1,\Omega} + \|\zeta\|_{0,\Omega} \leq C\Big(\|\mathbf{r}\|_{\mathbf{H}^{-1}} + \|m\|_{0,\Omega} + \|g\|_{-1,\Omega}\Big). \tag{4.3}$$

As consequence of regularity theory of partial differential equation (see [4,10,21]), if $(\mathbf{r}, m, g) \in [\mathbf{L}^2(\Omega)]^n \times L^2(\Omega) \times L^2(\Omega)$ we can also get that

$$\|\mathbf{v}\|_{\mathbf{H}^2} + \|\varrho\|_{2,\Omega} + \|\zeta\|_{1,\Omega} \leq C\Big(\|\mathbf{r}\|_{\mathbf{L}^2} + \|m\|_{0,\Omega} + \|g\|_{0,\Omega}\Big). \tag{4.4}$$

Furthermore, before obtaining the a posteriori error estimates for the states and co-states, we firstly give some useful lemmas.

Lemma 4.1. *Let I_h be the standard Lagrange interpolation operator. For $m = 0$ or 1, $q > \frac{n}{2}$ and $v \in W^{2,q}(\Omega)$,*

$$|v - I_h v|_{W^{m,q}(\Omega^h)} \leq C h^{2-m} |v|_{W^{2,q}(\Omega^h)}. \tag{4.5}$$

Lemma 4.2. *Let π_h be the average interpolation operator defined in [27], then there is a constant C such that*

$$|v - \pi_h v|_{m,p,\tau} \leq C \sum_{\bar{\tau}' \cap \bar{\tau} \neq \emptyset} h_\tau^{1-m} |v|_{1,p,\tau'},$$

for $v \in W^{1,p}(\Omega^h)$, $1 \leq p \leq \infty$, and $m = 0$ or 1.

Lemma 4.3. ([13]) *There is a constant C such that for all $v \in W^{1,p}(\Omega^h)$, $1 \le p < \infty$, the following inequality hold*

$$\|v\|_{W^{0,p}(\partial\tau)} \le C\big(h_\tau^{-\frac{1}{p}}\|v\|_{W^{0,p}(\tau)} + h_\tau^{1-\frac{1}{p}}|v|_{W^{1,p}(\tau)}\big).$$

Moreover, let us recall the Raviart-Thomas projection $\Pi_h : \mathbf{V} \to \mathbf{V}_h$, which satisfies: for any $\mathbf{v} \in \mathbf{V}$

$$(div(\mathbf{v} - \Pi_h\mathbf{v}), w_h) = 0, \quad \forall\, w_h \in X_h.$$

Then, we also know that $div\Pi_h = P_h div : V \to W_h$, and the following approximation properties:

$$\begin{aligned} &\|\mathbf{v} - \Pi_h\mathbf{v}\|_{\mathbf{L^2}} \le Ch\|\mathbf{v}\|_{\mathbf{H^1}} && for\ \mathbf{v} \in \mathbf{V},\\ &\|div(\mathbf{v} - \Pi_h\mathbf{v})\|_{0,\Omega} \le Ch\|div\mathbf{v}\|_{1,\Omega} && for\ div\mathbf{v} \in H^1. \end{aligned}$$

Moreover, in order to derive sharp a posteriori error estimates, we divide Ω into some subsets:

$$\begin{aligned} \Omega_d^- &= \{x \in \Omega :\ \varphi_h \le -\alpha d\},\\ \Omega_d &= \{x \in \Omega :\ \varphi_h > -\alpha d,\ Q_h = d\},\\ \Omega_d^+ &= \{x \in \Omega :\ \varphi_h > -\alpha d,\ Q_h > d\}. \end{aligned}$$

Then, it is easy to see that above three subsets are not intersected each other, and

$$\bar\Omega = \bar\Omega_d^- \cup \bar\Omega_d \cup \bar\Omega_d^+.$$

Remark 4.1. In the sequential, we fixed a discretized solution which converges to a related nonsingular solution of our system, and which means that we give the a posteriori error estimates of the pairs of local solutions under above assumptions.

Now let us have an intuitive analysis on the approximation error for the control. On Ω_d, asymptotically we can assume that

$$0 < \varphi_h + \alpha Q_h \to \varphi + \alpha Q.$$

Hence it follows from the optimality conditions that $Q = Q_h = d$ on Ω_d. Thus the error on Ω_d may be negligible. We should only need to estimate the error on

$$\Omega \setminus \Omega_d = \Omega_d^- \cup \Omega_d^+$$

in order to avoid over-estimate.

Hereafter, introduce

$$e^2 = \int_{\Omega_*} \big(\alpha Q + \varphi - \mathcal{R}_h(\alpha Q + \varphi)\big)^2,$$

$\mathcal{R}_h$ is the L^2-project operator from $L^2(\Omega)$ to K_h', and

$$\Omega_* = \Big\{x \in \Omega_d^+ :\ Q(x) = d,\ Q_h(x) > d\Big\}.$$

Furthermore, we introduce auxiliary functions $(\mathbf{u}(Q_h), p(Q_h), T(Q_h), \mathbf{w}(Q_h), \sigma(Q_h), \varphi(Q_h))$ satisfying the following problem:

$$\begin{aligned} &(a)\quad a_0(\mathbf{u}(Q_h), \mathbf{v}) + c_0(\mathbf{u}(Q_h), \mathbf{u}(Q_h), \mathbf{v}) + b(\mathbf{v}, p(Q_h))\\ &\qquad\qquad = d(T(Q_h), \mathbf{v}) + (f, \mathbf{v}) \quad \forall\, \mathbf{v} \in \mathbf{H}_0^1(\Omega),\\ &(b)\quad b(\mathbf{u}(Q_h), q) = 0 \quad \forall\, q \in L_0^2(\Omega), \qquad\qquad (4.6)\\ &(c)\quad a_1(T(Q_h), S) + c_1(\mathbf{u}(Q_h), T(Q_h), S) = (Q_h, S) \quad \forall\, S \in H_0^1(\Omega) \end{aligned}$$

and

$$
\begin{aligned}
&(a)\quad a_0(\mathbf{w}(Q_h),\mathbf{v})+c_0(\mathbf{v},\mathbf{u}(Q_h),\mathbf{w}(Q_h))+c_0(\mathbf{u}(Q_h),\mathbf{v},\mathbf{w}(Q_h))-b(\mathbf{v},\sigma(Q_h))\\
&\qquad=(\mathbf{u}(Q_h)-\mathbf{U},\mathbf{v})-c_1(\mathbf{v},T(Q_h),\varphi(Q_h))\quad\forall\,\mathbf{v}\in\mathbf{H}_0^1(\Omega),\\
&(b)\quad b(\mathbf{w}(Q_h),q)=0\quad\forall\,q\in L_0^2(\Omega),\\
&(c)\quad a_1(\varphi(Q_h),S)+c_1(\mathbf{u}(Q_h),S,\varphi(Q_h))=d(S,\mathbf{w}(Q_h))\quad\forall\,S\in H_0^1(\Omega).
\end{aligned}
\tag{4.7}
$$

Then from the result of [15], we have the regularity:

$$
\begin{aligned}
&\|\mathbf{u}(Q_h)\|_{\mathbf{H}^2}+\|\mathbf{w}(Q_h)\|_{\mathbf{H}^2}+\|T(Q_h)\|_{H^2}+\|\varphi(Q_h)\|_{H^2}+\|p(Q_h)\|_{H^1}+\|\sigma(Q_h)\|_{H^1}\\
&\le C\Big(\|f\|_{L^2}+\|Q_h\|_{L^2}\Big).
\end{aligned}
$$

Lemma 4.4. *Let Q and Q_h be the solutions of* (1.1) *and* (3.2) *respectively. Based on the above convexity assumption* (4.1), *then for sufficient small h*

$$
e^2+\|Q-Q_h\|_{0,\Omega}^2\le C\big(\eta_1^2+\|\varphi(Q_h)-\varphi_h\|_{0,\Omega}^2\big),\tag{4.8}
$$

where φ_h and $\varphi(Q_h)$ are the solutions of the equations (3.5) *and* (4.7) *respectively, and*

$$
\eta_1^2=\int_{\Omega_d^-\cup\Omega_d^+}(\alpha Q_h+\varphi_h)^2.
$$

Proof. It follows from the assumption (4.1) that

$$
\begin{aligned}
&c_*\|Q-Q_h\|_{L^2}^2\le (J'(Q),Q-Q_h)-(J'(Q_h),Q-Q_h)\\
&\le -(J'(Q_h),Q-Q_h)=(J_h'(Q_h),Q_h-Q)+(J_h'(Q_h)-J'(Q_h),Q-Q_h).
\end{aligned}
\tag{4.9}
$$

Note that

$$
(J_h'(Q_h),Q_h-Q)=\int_{\Omega_d^-\cup\Omega_d^+}(\alpha Q_h+\varphi_h)(Q_h-Q)+\int_{\Omega_d}(\alpha d+\varphi_h)(d-Q),\tag{4.10}
$$

it follows from the Schwarz's inequality and the inequality $2ab\le a^2/\delta+\delta b^2$ that

$$
\begin{aligned}
&\int_{\Omega_d^-\cup\Omega_d^+}(\alpha Q_h+\varphi_h)(Q_h-Q)\\
&\le\frac{1}{2\delta}\int_{\Omega_d^-\cup\Omega_d^+}(\alpha Q_h+\varphi_h)^2+\frac{\delta}{2}\|Q_h-Q\|_{L^2}^2=\frac{1}{2\delta}\eta_1^2+\frac{\delta}{2}\|Q_h-Q\|_{L^2}^2,
\end{aligned}
\tag{4.11}
$$

where $\delta>0$ is a constant which will be specified later.

It follows from the definition of Ω_d that $(\alpha d+\varphi_h)>0$ on Ω_d. Because that $d-Q\le 0$, we have that

$$
\int_{\Omega_d}(\alpha d+\varphi_h)(d-Q)\le 0.\tag{4.12}
$$

It follows from (4.10)-(4.12) that

$$
(J_h'(Q_h),Q_h-Q)\le C(\delta)\eta_1^2+\delta\|Q_h-Q\|_{L^2}^2.\tag{4.13}
$$

By using the formulas of J', J_h', it follows that

$$\begin{aligned}
&(J_h'(Q_h) - J'(Q_h), Q - Q_h)\\
&= (\alpha Q_h + \varphi_h, Q - Q_h) - (\alpha Q_h + \varphi(Q_h), Q - Q_h)\\
&= (\varphi_h - \varphi(Q_h), Q - Q_h) \le \frac{1}{2\delta}\|\varphi_h - \varphi(Q_h)\|_{L^2}^2 + \frac{\delta}{2}\|Q_h - Q\|_{L^2}^2\\
&\le C(\delta)\|\varphi_h - \varphi(Q_h)\|_{L^2}^2 + \frac{\delta}{2}\|Q_h - Q\|_{L^2}^2. \qquad (4.14)
\end{aligned}$$

Note that $Q_h > d$ on Ω_*, then $\mathcal{R}_h(\alpha Q_h + \varphi_h) = 0$ on Ω_*, and the proof can be given similarly as in [14]. Therefore,

$$\begin{aligned}
e^2 &= \int_{\Omega_*} \big((\alpha Q + \varphi) - \mathcal{R}_h(\alpha Q + \varphi)\big)^2\\
&\le C\int_{\Omega_*} \big((\alpha Q + \varphi) - (\alpha Q_h + \varphi_h)\big)^2 + C\int_{\Omega_*} (\alpha Q_h + \varphi_h)^2\\
&\quad + C\int_{\Omega_*} \big(\mathcal{R}_h(\alpha Q_h + \varphi_h)\big)^2 + C\int_{\Omega_*} \big(\mathcal{R}_h(\alpha Q + \varphi) - \mathcal{R}_h(\alpha Q_h + \varphi_h)\big)^2\\
&\le C(\|\varphi - \varphi_h\|_{0,\Omega}^2 + \|Q - Q_h\|_{0,\Omega}^2) + C\int_{\Omega_*} (\alpha Q_h + \varphi_h)^2\\
&\le C\eta_1^2 + C\|\varphi(Q_h) - \varphi_h\|_{0,\Omega}^2 + \|\varphi - \varphi(Q_h)\|_{0,\Omega}^2. \qquad (4.15)
\end{aligned}$$

From the related result of [4, 10, 14, 15, 21] for sufficient small h, we can have

$$\|\varphi - \varphi(Q_h)\|_{0,\Omega} \le C\|Q - Q_h\|_{0,\Omega}.$$

Therefore, (4.8) follows from (4.9), (4.13), (4.14) (4.15) by setting $\delta = \frac{c_*}{3}$. □

In the following parts, we give the main results of this paper. Before that, let us to show some lemmas. The proof of our main result is completed by the following lemmas.

Lemma 4.5. *Let $(\boldsymbol{u}(Q_h), T(Q_h), p(Q_h))$ and $(\boldsymbol{u}_h, T_h, p_h)$ be the solutions of* (3.4) *and* (4.6) *respectively. Let $(\boldsymbol{w}(Q_h), \varphi(Q_h), \sigma(Q_h))$ and $(\boldsymbol{w}_h, \varphi_h, \sigma_h)$ be the solutions of the co-state equations* (3.5) *and* (4.7) *respectively. Suppose that above assumptions are fulfilled, then for sufficient small h, we have the following estimate*

$$\|\boldsymbol{u}(Q_h) - \boldsymbol{u}_h\|_{\boldsymbol{L}^2(\Omega)}^2 + \|T(Q_h) - T_h\|_{0,\Omega}^2 + \|p(Q_h) - p_h\|_{0,\Omega}^2 \le C\sum_{i=2}^{5}\eta_i^2, \qquad (4.16)$$

where

$$\begin{aligned}
\eta_2^2 &= \sum_{\tau\in\mathcal{T}^h}\int_\tau h_\tau^4\big| -\nu\Delta\mathbf{u}_h + (\mathbf{u}_h\cdot\nabla)\mathbf{u}_h - T_h\mathbf{g} + \nabla p_h - f\big|^2,\\
\eta_3^2 &= \sum_{l\cap\partial\Omega=\emptyset}\int_l h_l^3[\nu\nabla\mathbf{u}_h\cdot n - p_h\mathbf{n}]^2, \quad \eta_4^2 = \sum_{\tau\in\mathcal{T}^h}\int_\tau h_\tau^4\big| -\kappa\Delta T_h + \mathbf{u}_h\cdot\nabla T_h - Q_h\big|^2,\\
\eta_5^2 &= \sum_{l\cap\partial\Omega=\emptyset}\int_l h_l^3[\kappa\nabla T_h\cdot n]^2,
\end{aligned}$$

with l is a face of an element τ, $[\nu\nabla\mathbf{u}_h\cdot\mathbf{n}]$ and $[\kappa\nabla T_h\cdot n]$ are the normal derivative jumps over the interior face l, defined by

$$[\nu\nabla\mathbf{u}_h\cdot\mathbf{n}]_l=(\nu\nabla\mathbf{u}_h|_{\tau_l^1}-\nu\nabla\mathbf{u}_h|_{\tau_l^2})\cdot\mathbf{n},$$
$$[\kappa\nabla T_h\cdot n]_l=(\kappa\nabla T_h|_{\tau_l^1}-\kappa\nabla T_h|_{\tau_l^2})\cdot n,$$

where n is the unit normal vector on $l=\bar{\tau}_l^1\cap\bar{\tau}_l^2$ outwards τ_l^1, h_l is the maximum diameter of the face l.

Proof. First, we introduce the following system:

$$\begin{aligned}
&(a)\quad -\nu\Delta\mathbf{R}-(\mathbf{u}(Q_h)\cdot\nabla)\mathbf{R}+\nabla\mathbf{u}(Q_h)^{tr}\mathbf{R}-\nabla\lambda+\phi\nabla T(Q_h)=\mathbf{u}(Q_h)-\mathbf{u}_h\quad in\ \Omega,\\
&(b)\quad \nabla\cdot\mathbf{R}=p(Q_h)-p_h\quad in\ \Omega,\\
&(c)\quad -\kappa\Delta\phi-\mathbf{u}(Q_h)\cdot\nabla\phi-\mathbf{R}\cdot\mathbf{g}=T(Q_h)-T_h\quad in\ \Omega,\\
&(d)\quad \mathbf{R}=0\quad \phi=0\quad on\ \partial\Omega.
\end{aligned}\tag{4.17}$$

Because we assume the solution $(\mathbf{u},p,T)$ is regular, so the linear system (4.17) is uniquely solvable and satisfies the a priori estimate

$$\begin{aligned}
&\|\mathbf{R}\|_{\mathbf{H}^2(\Omega)}+\|\lambda\|_{1,\Omega}+\|\phi\|_{2,\Omega}\\
&\le C\Big(\|\mathbf{u}(Q_h)-\mathbf{u}_h\|_{\mathbf{L}^2(\Omega)}+\|p(Q_h)-p_h\|_{0,\Omega}+\|T(Q_h)-T_h\|_{0,\Omega}\Big).
\end{aligned}\tag{4.18}$$

Next, let us denote $\xi=\mathbf{u}(Q_h)-\mathbf{u}_h$, $\eta=T(Q_h)-T_h$ and $\zeta=p(Q_h)-p_h$. Note that (3.1) and $b(\mathbf{u}_h,q_h)=0$, then we see $\nabla\cdot\mathbf{u}_h=0$. From Lemmas 4.1, 4.2, 4.3 and using the well known residual techniques we have

$$\begin{aligned}
&\|\xi\|^2+\|\eta\|^2+\|\zeta\|^2\\
&=a_0(\mathbf{u}(Q_h)-\mathbf{u}_h,\mathbf{R})+c_0(\mathbf{u}(Q_h),\mathbf{u}(Q_h),\mathbf{R})-c_0(\mathbf{u}_h,\mathbf{u}_h,\mathbf{R})-b(\mathbf{u}(Q_h)-\mathbf{u}_h,\lambda)\\
&\quad-d(T_h(Q)-T_h,\mathbf{R})+b(\mathbf{R},p(Q_h)-p_h+a_1(T(Q_h)-T_h,\phi)\\
&\quad+c_1(\mathbf{u}(Q_h),T(Q_h),\phi)-c_1(\mathbf{u}_h,T_h,\phi)-c_0(\xi,\xi,\mathbf{R})-c_1(\xi,\eta,\phi)\\
&=(f,\mathbf{R}-I_h\mathbf{R})-a_0(\mathbf{u}_h,\mathbf{R}-I_h\mathbf{R})-c_0(\mathbf{u}_h,\mathbf{u}_h,\mathbf{R}-I_h\mathbf{R})-b(\mathbf{R}-I_h\mathbf{R},p_h)\\
&\quad+d(T_h,\mathbf{R}-I_h\mathbf{R})-a_1(T_h,\phi-I_h\phi)-c_1(\mathbf{u}_h,T_h,\phi-I_h\phi)\\
&\quad+(Q_h,\phi-I_h\phi)-c_0(\xi,\xi,\mathbf{R})-c_1(\xi,\eta,\phi)\\
&=\sum_{\tau\in\mathcal{T}^h}\int_\tau\big(f+\nu\Delta\mathbf{u}_h-(\mathbf{u}_h\cdot\nabla)\mathbf{u}_h+T_h\mathbf{g}-\nabla p_h\big)\big(\mathbf{R}-I_h\mathbf{R}\big)\\
&\quad+\sum_{\tau\in\mathcal{T}^h}\int_{\partial\tau}\big(\nu\nabla\mathbf{u}_h\cdot n-p_h\mathbf{n}\big)\big(\mathbf{R}-I_h\mathbf{R}\big)ds+\sum_{\tau\in\mathcal{T}^h}\int_\tau\big(Q_h+\kappa\Delta T_h-\mathbf{u}_h\cdot\nabla T_h\big)\big(\phi-I_h\phi\big)\\
&\quad+\sum_{\tau\in\mathcal{T}^h}\int_{\partial\tau}\big(\nabla T_h\cdot n\big)\big(\phi-I_h\phi\big)ds-c_0(\xi,\xi,\mathbf{R})-c_1(\xi,\eta,\phi)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{\tau\in\mathcal{T}^h}\int_\tau \Big(f+\nu\Delta\mathbf{u}_h-(\mathbf{u}_h\cdot\nabla)\mathbf{u}_h+T_h\mathbf{g}-\nabla p_h\Big)\Big(\mathbf{R}-I_h\mathbf{R}\Big)\\
&\quad+\sum_{l\cap\partial\Omega=\emptyset}\int_l[\nu\nabla\mathbf{u}_h\cdot n-p_h\mathbf{n}](\mathbf{R}-I_h\mathbf{R})ds+\sum_{\tau\in\mathcal{T}^h}\int_\tau\big(Q_h+\kappa\Delta T_h-\mathbf{u}_h\cdot\nabla T_h\big)\big(\phi-I_h\phi\big)\\
&\quad+\sum_{l\cap\partial\Omega=\emptyset}\int_l[\nabla T_h\cdot n](\phi-I_h\phi)ds-c_0(\xi,\xi,\mathbf{R})-c_1(\xi,\eta,\phi).
\end{aligned}
\tag{4.19}
$$

Consequently, we have

$$
\begin{aligned}
&\|\xi\|^2+\|\eta\|^2+\|\zeta\|^2\\
\le\;& C(\delta)\sum_{\tau\in T^h}h_\tau^4\int_\tau\big|f+\nu\Delta\mathbf{u}_h-(\mathbf{u}_h\cdot\nabla)\mathbf{u}_h+T_h\mathbf{g}-\nabla p_h\big|^2\\
&+C(\delta)\sum_{l\cap\partial\Omega=\emptyset}h_l^3\int_l[\nu\nabla\mathbf{u}_h\cdot n-p_h\mathbf{n}]^2\\
&+C(\delta)\sum_{\tau\in T^h}h_\tau^4\int_\tau\big|Q_h+\kappa\Delta T_h-\mathbf{u}_h\cdot\nabla T_h\big|^2+C(\delta)\sum_{l\cap\partial\Omega=\emptyset}h_l^3\int_l[\kappa\nabla T_h\cdot n]^2\\
&+C\|\xi\|\big(\|\mathbf{R}\|_{\mathbf{H}^2}+\|\phi\|_{H^2}\big)\big(\|\xi\|_{\mathbf{H}^1}+\|\eta\|_{H^1}\big)+\delta\big(\|\mathbf{R}\|^2_{\mathbf{H}^2}+\|\phi\|^2_{H^2}+\|\lambda\|^2_{H^1}\big)\\
\le\;& C(\delta)\sum_{i=2}^5\eta_i^2+\delta\big(\|\xi\|^2+\|\eta\|^2+\|\zeta\|^2\big)+C\big(\|\xi\|^2+\|\eta\|^2+\|\zeta\|^2\big)\big(\|\xi\|_{\mathbf{H}^1}+\|\eta\|_{H^1}\big).
\end{aligned}
$$

By the Remark 4.1, we can have $\|\xi\|_{\mathbf{H}^1}+\|\eta\|_{H^1}+\|\zeta\|\to 0$ when $h\to 0$. So if we choose sufficient small h and δ, then $\big(\|\xi\|^2+\|\eta\|^2+\|\zeta\|^2\big)\big(\|\xi\|_{\mathbf{H}^1}+\|\eta\|_{H^1}\big)$ would be much less than $\|\xi\|^2+\|\eta\|^2+\|\zeta\|^2$. Moreover, we can prove the estimate (4.16). □

Next, we will give the H^1-norm estimates.

Lemma 4.6. *Let* $(\boldsymbol{u}(Q_h),T(Q_h),p(Q_h))$ *and* $(\boldsymbol{u}_h,T_h,p_h)$ *be the solutions of* (3.4) *and* (4.6) *respectively. Let* $(\boldsymbol{w}(Q_h),\varphi(Q_h),\sigma(Q_h))$ *and* $(\boldsymbol{w}_h,\varphi_h,\sigma_h)$ *be the solutions of the co-state equations* (3.5) *and* (4.7) *respectively. Suppose above assumptions are fulfilled, then for sufficient small* h*, we have the following estimate*

$$
\|\boldsymbol{u}(Q_h)-\boldsymbol{u}_h\|^2_{\boldsymbol{H}^1(\Omega)}+\|T(Q_h)-T_h\|^2_{1,\Omega}\le C\sum_{i=2}^{9}\eta_i^2,
\tag{4.20}
$$

$$
\|\boldsymbol{w}(Q_h)-\boldsymbol{w}_h\|^2_{\boldsymbol{H}^1(\Omega)}+\|\varphi(Q_h)-\varphi_h\|^2_{1,\Omega}\le C\sum_{i=2}^{13}\eta_i^2,
\tag{4.21}
$$

where

$$
\begin{aligned}
\eta_6^2&=\sum_{\tau\in\mathcal{T}^h}\int_\tau h_\tau^2\big|f+\nu\Delta\mathbf{u}_h-(\mathbf{u}_h\cdot\nabla)\mathbf{u}_h-\nabla p_h+T_h\mathbf{g}\big|^2,\\
\eta_7^2&=\sum_{l\cap\partial\Omega=\emptyset}\int_l h_l[\nu\nabla\mathbf{u}_h\cdot n-p_h\mathbf{n}]^2,\quad \eta_8^2=\sum_{\tau\in\mathcal{T}^h}\int_\tau h_\tau^2\big|\kappa\Delta T_h-\mathbf{u}_h\cdot\nabla T_h+Q_h\big|^2,\\
\eta_9^2&=\sum_{l\cap\partial\Omega=\emptyset}\int_l h_l[\kappa\nabla T_h\cdot n]^2,\\
\eta_{10}^2&=\sum_{\tau\in\mathcal{T}^h}\int_\tau h_\tau^2\big|\boldsymbol{u}_h-\boldsymbol{U}+\nu\Delta\mathbf{w}_h+(\mathbf{u}_h\cdot\nabla)\mathbf{w}_h-\boldsymbol{w}_h\cdot\nabla\mathbf{u}_h+\nabla\sigma_h+\varphi_h\nabla T_h\big|^2,
\end{aligned}
$$

$$\eta_{11}^2 = \sum_{l\cap\partial\Omega=\emptyset}\int_l h_l[\nu\nabla\mathbf{w}_h\cdot n+\sigma_h\mathbf{n}]^2,\quad \eta_{12}^2=\sum_{\tau\in\mathcal{T}^h}\int_\tau h_\tau^2\big|\kappa\Delta\varphi_h+\mathbf{u}_h\cdot\nabla\varphi_h+\boldsymbol{w}_h\cdot\mathbf{g}\big|^2,$$
$$\eta_{13}^2 = \sum_{l\cap\partial\Omega=\emptyset}\int_l h_l[\kappa\nabla\varphi_h\cdot n]^2,$$

with l is a face of an element τ, $[\nu\nabla\mathbf{u}_h\cdot n]$ and $[\kappa\nabla T_h\cdot n]$ are the normal derivative jumps over the interior face l, defined by

$$[\nu\nabla\mathbf{w}_h\cdot n]_l = (\nu\nabla\mathbf{w}_h|_{\tau_l^1}-\nu\nabla\mathbf{w}_h|_{\tau_l^2})\cdot n,$$

$$[\kappa\nabla\varphi_h\cdot n]_l = (\kappa\nabla\varphi_h|_{\tau_l^1}-\kappa\nabla\varphi_h|_{\tau_l^2})\cdot n,$$

where n is the unit normal vector on $l=\bar{\tau}_l^1\cap\bar{\tau}_l^2$ outwards τ_l^1, h_l is the maximum diameter of the face l.

Proof. From equations (3.4) and (4.6) and adopt the same definition of ξ,η as in Lemma 4.5, we can have

$$\begin{aligned}
&a_0(\mathbf{u}(Q_h)-\mathbf{u}_h,\mathbf{v})+c_0(\mathbf{u}_h,\mathbf{u}(Q_h)-\mathbf{u}_h,\mathbf{v})\\
&= d(T(Q_h)-T_h,\mathbf{v})+(f,\mathbf{v})-c_0(\mathbf{u}(Q_h)-\mathbf{u}_h,\mathbf{u}(Q_h),\mathbf{v})-b(\mathbf{v},p(Q_h)-p_h)\\
&\quad -a_0(\mathbf{u}_h,\mathbf{v})-c_0(\mathbf{u}_h,\mathbf{u}_h,\mathbf{v})-b(\mathbf{v},p_h)-d(T_h,\mathbf{v})\quad\forall\ \mathbf{v}\in\mathbf{H}_0^1(\Omega),\\
&b(\mathbf{u}(Q_h)-\mathbf{u}_h,q)=-b(\mathbf{u}_h,q)\quad\forall\ q\in L_0^2(\Omega),\\
&a_1(T(Q_h)-T_h,S)+c_1(\mathbf{u}_h,T(Q_h)-T_h,S)\\
&=(Q_h,S)-c_1(\mathbf{u}(Q_h)-\mathbf{u}_h,T(Q_h),S)-a_1(T_h,S)-c_1(\mathbf{u}_h,T_h,S)\quad\forall\ S\in H_0^1(\Omega).
\end{aligned}\tag{4.22}$$

Now, from (3.1) and choosing $\mathbf{v}=\xi,\ S=\eta$, we can have that

$$\nabla\cdot\mathbf{u}_h=0,\quad \nabla\cdot\mathbf{w}_h=0,\quad c_0(\mathbf{u}_h,\xi,\xi)=0,\quad c_1(\mathbf{u}_h,\eta,\eta)=0.$$

Furthermore, let Π_h is the Raviart-Thomas Projection defined above. Then it gives that

$$\begin{aligned}
&c\big(\|\xi\|_{\mathbf{H}^1}^2+\|\eta\|_{H^1}^2\big)\le a_0(\xi,\xi)+a_1(\eta,\eta)\\
&=-c_0(\xi,\mathbf{u}(Q_h),\xi)-b(\xi,p(Q_h)-p_h)+d(\eta,\xi)+(f,\xi)-a_0(\mathbf{u}_h,\xi)-c_0(\mathbf{u}_h,\mathbf{u}_h,\xi)\\
&\quad +b(\xi,p_h)-d(T_h,\xi)-c_1(\xi,T(Q_h),\eta)-a_1(T_h,\eta)-c_1(\mathbf{u}_h,T_h,\eta)+(Q_h,\eta)\\
&=-c_0(\xi,\mathbf{u}(Q_h),\xi)-b(\xi,p(Q_h)-p_h)-c_1(\xi,T(Q_h),\eta)+d(\eta,\xi)+(f,\xi-\Pi_h\xi)\\
&\quad -a_0(\mathbf{u}_h,\xi-\Pi_h\xi)-c_0(\mathbf{u}_h,\mathbf{u}_h,\xi-\Pi_h\xi)+b(\xi-\Pi_h\xi,p_h)-d(T_h,\xi-\Pi_h\xi)\\
&\quad -a_1(T_h,\eta-\pi_h\eta)-c_1(\mathbf{u}_h,T_h,\eta-\pi_h\eta)+(Q_h,\eta-\pi_h\eta)\\
&=\sum_{\tau\in\mathcal{T}^h}\int_\tau\big(f+\nu\Delta\mathbf{u}_h-(\mathbf{u}_h\cdot\nabla)\mathbf{u}_h-\nabla p_h+T_h\mathbf{g}\big)\big(\xi-\Pi_h\xi\big)\\
&\quad +\sum_{\tau\in\mathcal{T}^h}\int_{\partial\tau}\big(\nu\nabla\mathbf{u}_h\cdot n-p_h\mathbf{n}\big)\big(\xi-\Pi_h\xi\big)ds+\sum_{\tau\in\mathcal{T}^h}\int_\tau\big(Q_h+\kappa\Delta T_h-\mathbf{u}_h\cdot\nabla T_h\big)\big(\eta-\pi_h\eta\big)\\
&\quad +\sum_{\tau\in\mathcal{T}^h}\int_{\partial\tau}\big(\nabla T_h\cdot n\big)\big(\eta-\pi_h\eta\big)ds-c_0(\xi,\mathbf{u}(Q_h),\xi)\\
&\quad -b(\xi,p(Q_h)-p_h)-c_1(\xi,T(Q_h),\eta)+d(\eta,\xi).
\end{aligned}$$

Consequently, we have

$$
\begin{aligned}
&c\big(\|\xi\|_{\mathbf{H}^1}^2+\|\eta\|_{H^1}^2\big)\\
\le& C(\delta)\sum_{\tau\in T^h}h_\tau^2\int_\tau\big|f+\nu\Delta\mathbf{u}_h-(\mathbf{u}_h\cdot\nabla)\mathbf{u}_h-\nabla p_h+T_h\mathbf{g}\big|^2\\
&+C(\delta)\sum_{l\cap\partial\Omega=\emptyset}h_l\int_l[\nu\nabla\mathbf{u}_h\cdot n-p_h\mathbf{n}]^2+C(\delta)\sum_{\tau\in T^h}h_\tau^2\int_\tau\big|Q_h+\kappa\Delta T_h-\mathbf{u}_h\cdot\nabla T_h\big|^2\\
&+C(\delta)\sum_{l\cap\partial\Omega=\emptyset}h_l\int_l[\kappa\nabla T_h\cdot n]^2+C\big(\|\xi\|_{\mathbf{L}^2}^2+\|\eta\|_{L^2}^2\big)+\delta\big(\|\xi\|_{\mathbf{H}^1}^2+\|\eta\|_{H^1}^2\big)\\
\le& C(\delta)\sum_{i=2}^{9}\eta_i^2+\delta\big(\|\xi\|_{\mathbf{H}^1}^2+\|\eta\|_{H^1}^2\big).
\end{aligned}
$$

Now, choosing sufficient small δ, (4.20) follows. We also see that equations (3.5) and (4.7) lead to

$$
\begin{aligned}
&a_0(\mathbf{w}(Q_h)-\mathbf{w}_h,\mathbf{v})+c_0(\mathbf{v},\mathbf{u}(Q_h),\mathbf{w}(Q_h)-\mathbf{w}_h)+c_0(\mathbf{v},\mathbf{u}(Q_h)-\mathbf{u}_h,\mathbf{w}_h)\\
&\qquad+c_0(\mathbf{u}(Q_h),\mathbf{v},\mathbf{w}(Q_h)-\mathbf{w}_h)+c_0(\mathbf{u}(Q_h)-\mathbf{u}_h,\mathbf{v},\mathbf{w}_h)-b(\mathbf{v},\sigma(Q_h)-\sigma_h)\\
&\quad=(\mathbf{u}(Q_h)-\mathbf{U},\mathbf{v})+c_1(\mathbf{v},T(Q_h),\varphi(Q_h)-\varphi_h)+c_1(\mathbf{v},T(Q_h)-T_h,\varphi_h)\\
&\qquad-a_0(\mathbf{w}_h,\mathbf{v})-c_0(\mathbf{v},\mathbf{u}_h,\mathbf{w}_h)-c_0(\mathbf{u}_h,\mathbf{v},\mathbf{w}_h)+b(\mathbf{v},\varphi_h)\\
&\qquad-c_0(\mathbf{v},T_h,\varphi_h)+(\mathbf{u}_h-\mathbf{U},\mathbf{v})\quad\forall\ \mathbf{v}\in\mathbf{H}_0^1(\Omega),\\
&b(\mathbf{w}(Q_h)-\mathbf{w}_h,q)=-b(\mathbf{w}_h,q)\quad\forall\ q\in L_0^2(\Omega),\\
&a_1(\varphi(Q_h)-\varphi_h,S)+c_1(\mathbf{u}(Q_h)-\mathbf{u}_h,S,\varphi_h)+c_1(\mathbf{u}(Q_h),S,\varphi(Q_h)-\varphi_h)\\
&\quad=d(\mathbf{w}(Q_h)-\mathbf{w}_h,S)-a_1(\varphi_h,S)-c_1(\mathbf{u}_h,S,\varphi_h)+d(\mathbf{w}_h,S)\\
&\qquad-c_1(\mathbf{u}(Q_h)-\mathbf{u}_h,T(Q_h),S)-a_1(T_h,S)-c_1(\mathbf{u}_h,T_h,S)\quad\forall\ S\in H_0^1(\Omega).
\end{aligned}
\tag{4.23}
$$

Similarly, denoting $\xi^*=\mathbf{w}(Q_h)-\mathbf{w}_h$, $\eta^*=\varphi(Q_h)-\varphi_h$ and $\zeta^*=\sigma(Q_h)-\sigma_h$, gives

$$
\begin{aligned}
&c\big(\|\xi^*\|_{\mathbf{H}^1}^2+\|\eta^*\|_{H^1}^2\big)\le a_0(\xi^*,\xi^*)+a_1(\eta^*,\eta^*)\\
=&-c_0(\xi^*,\mathbf{u}(Q_h),\xi^*)-c_0(\xi^*,\xi,\mathbf{u}_h)-c_0(\xi,\xi^*,\mathbf{w}_h)+b(\xi^*,\zeta^*)\\
&+(\xi,\xi^*)+c_1(\xi^*,T(Q_h),\eta^*)+c_1(\xi^*,\eta,\varphi_h)\\
&-a_0(\mathbf{w}_h,\xi^*)-c_0(\xi^*,\mathbf{u}_h,\mathbf{w}_h)-c_0(\mathbf{u}_h,\xi^*,\mathbf{w}_h)+b(\xi^*,\sigma_h)-c_1(\xi^*,T_h,\varphi_h)+(\mathbf{u}_h-\mathbf{U},\xi^*)\\
&-c_1(\xi,\eta,\varphi_h)+d(\xi^*,\eta^*)-a_1(\varphi_h,\eta^*)-c_1(\mathbf{u}_h,\eta^*,\varphi_h)+d(\mathbf{w}_h,\eta^*)\\
\le& C(\delta)\sum_{\tau\in T^h}h_\tau^2\int_\tau\Big|\mathbf{u}_h-\mathbf{U}+\nu\Delta\mathbf{w}_h+(\mathbf{u}_h\cdot\nabla)\mathbf{w}_h-\mathbf{w}_h\cdot\nabla\mathbf{u}_h+\nabla\sigma_h+\varphi_h\nabla T_h\Big|^2\\
&+C(\delta)\sum_{l\cap\partial\Omega=\emptyset}h_l\int_l\Big[\nu\nabla\mathbf{w}_h\cdot n+\sigma_h\mathbf{n}\Big]^2+C(\delta)\sum_{l\cap\partial\Omega=\emptyset}h_l\int_l\Big[\kappa\nabla T_h\cdot n\Big]^2\\
&+C(\delta)\sum_{\tau\in T^h}h_\tau^2\int_\tau\Big|\kappa\Delta T_h+\mathbf{u}_h\cdot\nabla\varphi_h+\mathbf{w}_h\cdot\mathbf{g}\Big|^2\\
&+C\Big(\|\xi\|_{\mathbf{H}^1}^2+\|\eta\|_{H^1}^2\Big)+\delta\Big(\|\xi^*\|_{\mathbf{H}^1}^2+\|\eta^*\|_{H^1}^2\Big)\\
\le& C(\delta)\sum_{i=2}^{13}\eta_i^2+\delta\Big(\|\xi^*\|_{\mathbf{H}^1}^2+\|\eta^*\|_{H^1}^2\Big).
\end{aligned}
$$

Then, (4.21) is obtained. $\square$

Lemma 4.7. *Let $(\boldsymbol{u}(Q_h), T(Q_h), p(Q_h))$ and $(\boldsymbol{u}_h, T_h, p_h)$ be the solutions of (3.4) and (4.6) respectively. Let $(\boldsymbol{w}(Q_h), \varphi(Q_h), \sigma(Q_h))$ and $(\boldsymbol{w}_h, \varphi_h, \sigma_h)$ be the solutions of the co-state equations (3.5) and (4.7) respectively. Suppose above assumptions are fulfilled, then for sufficient small h, we have the following estimate*

$$\begin{aligned}&\|\boldsymbol{w}(Q_h)-\boldsymbol{w}_h\|^2_{\boldsymbol{L}^2(\Omega)}+\|\varphi(Q_h)-\varphi_h\|^2_{0,\Omega}+\|\sigma(Q_h)-\sigma_h\|^2_{0,\Omega}\\ \le& C\sum_{i=2}^{9}\eta_i^2+C\sum_{i=14}^{17}\eta_i^2,\end{aligned}\tag{4.24}$$

where

$$\begin{aligned}\eta_{14}^2=&\sum_{\tau\in\mathcal{T}^h}\int_\tau h_\tau^4\Big|\boldsymbol{u}_h-\boldsymbol{U}+\nu\Delta\mathbf{w}_h+(\mathbf{u}_h\cdot\nabla)\mathbf{w}_h-\boldsymbol{w}_h\cdot\nabla\mathbf{u}_h+\nabla\sigma_h+\varphi_h\nabla T_h\Big|^2,\\ \eta_{15}^2=&\sum_{l\cap\partial\Omega=\emptyset}\int_l h_l^3\Big[\nu\nabla\mathbf{w}_h\cdot n+\sigma_h\mathbf{n}\Big]^2,\quad \eta_{16}^2=\sum_{\tau\in\mathcal{T}^h}\int_\tau h_\tau^4\Big|\kappa\Delta\varphi_h+\mathbf{u}_h\cdot\nabla\varphi_h+\boldsymbol{w}_h\cdot\mathbf{g}\Big|^2,\\ \eta_{17}^2=&\sum_{l\cap\partial\Omega=\emptyset}\int_l h_l^3\Big[\kappa\nabla\varphi_h\cdot n\Big]^2.\end{aligned}$$

Proof. First, we introduce the following system:

$$\begin{aligned}&(a)\quad -\nu\Delta\mathbf{R}^*+(\mathbf{u}(Q_h)\cdot\nabla)\mathbf{R}^*+(\mathbf{R}^*\cdot\nabla)\mathbf{u}(Q_h)+\nabla\lambda^*-\phi^*\cdot\mathbf{g}\\ &\qquad\quad=\mathbf{w}(Q_h)-\mathbf{w}_h\quad in\ \Omega,\\ &(b)\quad -\nabla\cdot\mathbf{R}^*=\sigma(Q_h)-\sigma_h\quad in\ \Omega,\\ &(c)\quad -\kappa\Delta\phi^*+\mathbf{u}(Q_h)\cdot\nabla\phi^*+\mathbf{R}^*\cdot\nabla T(Q_h)=\varphi(Q_h)-\varphi_h\quad in\ \Omega,\\ &(d)\quad \mathbf{R}^*=0\quad \phi^*=0\quad on\ \partial\Omega.\end{aligned}\tag{4.25}$$

Because we assume the solution $(\mathbf{u}, p, T)$ is regular, and also the linear system (4.25) is the adjoint system of (4.17) so that it is uniquely solvable and satisfies the a priori estimate

$$\begin{aligned}&\|\mathbf{R}^*\|_{\mathbf{H}^2(\Omega)}+\|\lambda^*\|_{1,\Omega}+\|\phi^*\|_{2,\Omega}\\ \le& C\big(\|\mathbf{w}(Q_h)-\mathbf{w}_h\|_{\mathbf{L}^2(\Omega)}+\|\sigma(Q_h)-\sigma_h\|_{0,\Omega}+\|\varphi(Q_h)-\varphi_h\|_{0,\Omega}\big).\end{aligned}\tag{4.26}$$

Similarly, we can have

$$\begin{aligned}&\|\xi^*\|^2+\|\eta^*\|^2+\|\zeta^*\|^2\\ =&a_0(\mathbf{w}(Q_h)-\mathbf{w}_h,\mathbf{R}^*)+c_0(\mathbf{u}(Q_h),\mathbf{R}^*,\xi^*)+c_0(\mathbf{R}^*,\mathbf{u}_h,\xi^*)+b(\xi^*,\lambda^*)-d(\phi^*,\xi^*)\\ &+b(\mathbf{R}^*,\zeta^*)+a_1(\eta^*,\phi^*)+c_1(\mathbf{u}(Q_h),\phi^*,\eta^*)+c_1(\mathbf{R}^*,T(Q_h),\eta^*)\\ =&(\mathbf{u}_h-\mathbf{U},\mathbf{R}^*-I_h\mathbf{R}^*)-a_0(\mathbf{w}_h,\mathbf{R}^*-I_h\mathbf{R}^*)+c_0(\mathbf{u}_h,\mathbf{R}^*-I_h\mathbf{R}^*,\mathbf{w}_h)\\ &+C_0(\mathbf{R}^*-I_h\mathbf{R}^*,\mathbf{u}_h,\mathbf{w}_h)-b(\mathbf{R}^*-I_h\mathbf{R}^*,\sigma_h)+C_1(\mathbf{R}^*-I_h\mathbf{R}^*,T_h,\varphi_h)\\ &-a_1(\varphi_h,\phi^*-I_h\phi^*)-c_1(\mathbf{u}_h,\phi^*-I_h\phi^*,\varphi_h)-d(\phi^*-I_h\phi^*,\mathbf{w}_h)\\ &-c_0(\xi,\mathbf{R}^*,\mathbf{w}_h)-c_0(\mathbf{R}^*,\xi,\mathbf{w}_h)-c_1(\mathbf{R}^*,\eta,\phi_h)-c_1(\xi,\phi^*,\varphi_h)-(\xi,\mathbf{R}^*)\end{aligned}$$

$$
\begin{aligned}
\leq & C(\delta) \sum_{\tau\in\mathcal{T}^h} \int_\tau h_\tau^4 \Big|\mathbf{u}_h - \mathbf{U} + \nu\Delta\mathbf{w}_h + (\mathbf{u}_h\cdot\nabla)\mathbf{w}_h - \mathbf{w}_h\cdot\nabla\mathbf{u}_h + \nabla\sigma_h + \varphi_h\nabla T_h\Big|^2 \\
& + \sum_{l\cap\partial\Omega=\emptyset} \int_l h_l^3 \Big[\nu\nabla\mathbf{w}_h\cdot n + \sigma_h\mathbf{n}\Big]^2 + \sum_{l\cap\partial\Omega=\emptyset} \int_l h_l^3 \Big[\kappa\nabla\varphi_h\cdot n\Big]^2 \\
& + C(\delta) \sum_{\tau\in\mathcal{T}^h} \int_\tau h_\tau^4 \Big|\kappa\Delta T_h + \mathbf{u}_h\cdot\nabla\varphi_h + \mathbf{w}_h\cdot\mathbf{g}\Big|^2 \\
& + C\Big(\|\xi\|_{\mathbf{H}^1}^2 + \|\eta\|_{H^1}^2\Big) + \delta\Big(\|\mathbf{R}^*\|_{\mathbf{H}^2}^2 + \|\phi^*\|_{H^2}^2 + \|\lambda^*\|_{H^1}^2\Big) \\
\leq & C(\delta) \sum_{i=14}^{17} \eta_i^2 + C\Big(\|\xi\|_{\mathbf{H}^1}^2 + \|\eta\|_{H^1}^2\Big) + \delta\Big(\|\xi^*\|^2 + \|\eta^*\|^2 + \|\zeta^*\|^2\Big). \qquad (4.27)
\end{aligned}
$$

Hence, repeating the same arguments we have completed our proof. □

Next, we give our main result of this paper.

Theorem 4.1. *Let $(\boldsymbol{u}, T, p)$ and $(\boldsymbol{u}_h, T_h, p_h)$ be the solutions of* (2.4) *and* (4.6) *respectively. Let $(\boldsymbol{w}, \varphi, \sigma)$ and $(\boldsymbol{w}_h, \varphi_h, \sigma_h)$ be the solutions of the co-state equations* (2.5) *and* (4.7) *respectively. Suppose above assumptions are fulfilled, then for sufficient small h, we have the following estimate*

$$
\begin{aligned}
& e^2 + \|Q - Q_h\|_{0,\Omega}^2 + \|\boldsymbol{u} - \boldsymbol{u}_h\|_{\boldsymbol{H}^1(\Omega)}^2 + \|T - T_h\|_{H^1(\Omega)}^2 + \|p - p_h\|_{0,\Omega}^2 \\
& + \|\boldsymbol{w} - \boldsymbol{w}_h\|_{\boldsymbol{H}^1(\Omega)}^2 + \|\varphi - \varphi_h\|_{H^1(\Omega)}^2 + \|\sigma - \sigma_h\|_{0,\Omega}^2 \leq C\eta_1^2 + C\sum_{i=6}^{13} \eta_i^2,
\end{aligned} \qquad (4.28)
$$

Proof. As $(\mathbf{u}, T, p)$ is assumed to be a regular solution, for sufficient small h it gives

$$
\begin{aligned}
& \|\mathbf{u} - \mathbf{u}(Q_h)\|_{\mathbf{H}^1(\Omega)} + \|T - T(Q_h)\|_{H^1(\Omega)} + \|p - p(Q_h)\|_{0,\Omega} \leq C\|Q - Q_h\|_{0,\Omega}, \\
& \|\mathbf{w} - \mathbf{w}(Q_h)\|_{\mathbf{H}^1(\Omega)} + \|\varphi - \varphi(Q_h)\|_{H^1(\Omega)} + \|\sigma - \sigma(Q_h)\|_{0,\Omega} \leq C\|Q - Q_h\|_{0,\Omega}.
\end{aligned}
$$

Note that

$$
\|\mathbf{u} - \mathbf{u}_h\|_{\mathbf{H}^1(\Omega)} \leq \|\mathbf{u} - \mathbf{u}(Q_h)\|_{\mathbf{H}^1(\Omega)} + \|\mathbf{u}(Q_h) - \mathbf{u}_h\|_{\mathbf{H}^1(\Omega)}.
$$

Using the same technique to handle with other terms, then (4.28) follows from above lemmas. □

Now we are in the position to prove the a posteriori lower bound. In order to derive the a posteriori lower bound, we prove the following lemmas using the standard bubble function technique.

Lemma 4.8. *Let $(\boldsymbol{u}, T, p)$ and $(\boldsymbol{u}_h, T_h, p_h)$ be the solutions of* (2.4) *and* (4.6) *respectively.*

$$
\eta_6^2 + \eta_7^2 \leq C\|\boldsymbol{u} - \boldsymbol{u}_h\|_{\boldsymbol{H}^1(\Omega)}^2 + \|T - T_h\|_{H^1(\Omega)}^2 + \|p - p_h\|_{0,\Omega}^2 + C\epsilon_2^2, \qquad (4.29)
$$

where η_i is defined in Lemma 4.6*,*

$$
\epsilon_2^2 = \sum_{\tau\in\mathcal{T}^h} \int_\tau h_\tau^2 (f - \bar{f})^2,
$$

where

$$
\bar{v}|_\tau = \frac{\int_\tau v}{\int_\tau 1}.
$$

Proof. Using the the standard bubble function technique (see [25], for example), it can be proved that there exist polynomials $\mathbf{W}_\tau \in \mathbf{H}_0^1(\tau) \cap P_3$ and $\mathbf{W}_l \in \mathbf{H}_0^1(\tau_l^1 \cup \tau_l^2) \cap P_2$ such that

$$\begin{aligned}
&\int_\tau h_\tau^2 \Big| \bar{f} + \nu\Delta\mathbf{u}_h - (\mathbf{u}_h \cdot \nabla)\mathbf{u}_h - \nabla p_h + T_h\mathbf{g} \Big|^2 \\
=&\int_\tau \Big(\bar{f} + \nu\Delta\mathbf{u}_h - (\mathbf{u}_h \cdot \nabla)\mathbf{u}_h - \nabla p_h + T_h\mathbf{g} \Big)\mathbf{W}_\tau, && (4.30)\\
&\int_l h_l \Big[\nu\nabla\mathbf{u}_h \cdot n - p_h\mathbf{n} \Big]^2 = \int_l \Big[\nu\nabla\mathbf{u}_h \cdot n - p_h\mathbf{n} \Big]\mathbf{W}_l, && (4.31)\\
&\|\mathbf{W}_\tau\|_{\mathbf{H}^1(\tau)}^2 \le C \int_\tau h_\tau^2 \Big| \bar{f} + \nu\Delta\mathbf{u}_h - (\mathbf{u}_h \cdot \nabla)\mathbf{u}_h - \nabla p_h + T_h\mathbf{g} \Big|^2, && (4.32)\\
&h_\tau^{-2}\|\mathbf{W}_\tau\|_{L^2(\tau)}^2 \le C \int_\tau h_\tau^2 \Big| \bar{f} + \nu\Delta\mathbf{u}_h - (\mathbf{u}_h \cdot \nabla)\mathbf{u}_h - \nabla p_h + T_h\mathbf{g} \Big|^2, && (4.33)\\
&\|\mathbf{W}_l\|_{H^1(\tau_l^1 \cup \tau_l^2)}^2 \le C \int_l h_l \Big[\nu\nabla\mathbf{u}_h \cdot n - p_h\mathbf{n} \Big]^2, && (4.34)\\
&h_l^{-2}\|\mathbf{W}_l\|_{L^2(\tau_l^1 \cup \tau_l^2)}^2 \le C \int_l h_l \Big[\nu\nabla\mathbf{u}_h \cdot n - p_h\mathbf{n} \Big]^2. && (4.35)
\end{aligned}$$

Then, it follows from (4.30), (4.32), (4.33) and Schwartz inequality that

$$\begin{aligned}
&\int_\tau h_\tau^2 \Big| \bar{f} + \nu\Delta\mathbf{u}_h - (\mathbf{u}_h \cdot \nabla)\mathbf{u}_h - \nabla p_h + T_h\mathbf{g} \Big|^2 \\
=&\int_\tau \Big(\bar{f} + \nu\Delta\mathbf{u}_h - (\mathbf{u}_h \cdot \nabla)\mathbf{u}_h - \nabla p_h + T_h\mathbf{g} \Big)\mathbf{W}_\tau \\
=&\int_\tau \Big(f + \nu\Delta\mathbf{u}_h - (\mathbf{u}_h \cdot \nabla)\mathbf{u}_h - \nabla p_h + T_h\mathbf{g} \Big)\mathbf{W}_\tau + \int_\tau (\bar{f} - f)\mathbf{W}_\tau \\
\le&\int_\tau \Big(\nu\Delta\mathbf{u}_h - (\mathbf{u}_h \cdot \nabla)\mathbf{u}_h - \nabla p_h + T_h\mathbf{g} - (\nu\Delta\mathbf{u} - (\mathbf{u} \cdot \nabla)\mathbf{u} - \nabla p + T\mathbf{g}) \Big)\mathbf{W}_\tau \\
&+C\delta h_\tau^{-2}\|\mathbf{W}_\tau\|_{L^2(\tau)}^2 + C(\delta)\int_\tau h_\tau^2 \Big| \bar{f} - f \Big|^2 \\
=&-\int_\tau \nu\nabla(\mathbf{u}_h - \mathbf{u})\nabla\mathbf{W}_\tau - \int_\tau (p_h - p)\nabla \cdot \mathbf{W}_\tau + \int_\tau (T_h\mathbf{g} - T\mathbf{g})\mathbf{W}_\tau \\
&+\int_\tau \Big((\mathbf{u} \cdot \nabla)\mathbf{u} - (\mathbf{u}_h \cdot \nabla)\mathbf{u}_h \Big)\mathbf{W}_\tau + C\delta h_\tau^{-2}\|\mathbf{W}_\tau\|_{L^2(\tau)}^2 + C(\delta)\int_\tau h_\tau^2 \Big| \bar{f} - f \Big|^2 \\
\le& C(\delta)\Big(\|\mathbf{u} - \mathbf{u}_h\|_{\mathbf{H}^1(\tau)}^2 + \|T - T_h\|_{H^1(\tau)}^2 + \|p - p_h\|_{0,\tau}^2 \Big) \\
&+C\delta\Big(h_\tau^{-2}\|\mathbf{W}_\tau\|_{L^2(\tau)}^2 + \|\mathbf{W}_\tau\|_{H^1(\tau)}^2 \Big) + C(\delta)\int_\tau h_\tau^2 \Big| \bar{f} - f \Big|^2 \\
\le& C(\delta)\Big(\|\mathbf{u} - \mathbf{u}_h\|_{\mathbf{H}^1(\tau)}^2 + \|T - T_h\|_{H^1(\tau)}^2 + \|p - p_h\|_{0,\tau}^2 \Big) \\
&+C\delta\int_\tau h_\tau^2 \Big| \bar{f} + \nu\Delta\mathbf{u}_h - (\mathbf{u}_h \cdot \nabla)\mathbf{u}_h - \nabla p_h + T_h\mathbf{g} \Big|^2 + C(\delta)\int_\tau h_\tau^2 \Big| \bar{f} - f \Big|^2,
\end{aligned}$$

where δ is an arbitrary positive number. Therefore, letting $\delta = \frac{1}{2C}$ yields

$$\begin{aligned}
&\sum_\tau \int_\tau h_\tau^2 \big| \bar{f} + \nu\Delta\mathbf{u}_h - (\mathbf{u}_h \cdot \nabla)\mathbf{u}_h - \nabla p_h + T_h\mathbf{g} \big|^2 \\
\le& C(\delta)\big(\|\mathbf{u} - \mathbf{u}_h\|_{\mathbf{H}^1(\Omega)}^2 + \|T - T_h\|_{H^1(\Omega)}^2 + \|p - p_h\|_{0,\Omega}^2 + \epsilon_2^2 \big).
\end{aligned}$$

Similarly, when $l \cap \partial\Omega \neq \emptyset$, where $l = \bar{\tau}_l^1 \cap \bar{\tau}_l^2$, it follows from (4.31), (4.34) and (4.35) that

$$\begin{aligned}
&\int_l h_l \Big[\nu\nabla\mathbf{u}_h \cdot n - p_h\mathbf{n}\Big]^2 = \int_l \Big[\nu\nabla\mathbf{u}_h \cdot n - p_h\mathbf{n}\Big]\mathbf{W}_l \\
=&\int_l \Big[\nu\nabla\mathbf{u}_h \cdot n - p_h\mathbf{n} - (\nu\nabla\mathbf{u}\cdot n - p\mathbf{n})\Big]\mathbf{W}_l \\
=&\int_{\tau_l^1\cup\tau_l^2} \nu\nabla(\mathbf{u}_h - \mathbf{u})\nabla\mathbf{W}_l + \int_{\tau_l^1\cup\tau_l^2}(p-p_h)\nabla\cdot\mathbf{W}_l + \int_{\tau_l^1\cup\tau_l^2}\Big(\nu\Delta(\mathbf{u}_h-\mathbf{u}) + \nabla(p-p_h)\Big)\mathbf{W}_l \\
\le& C(\delta)\Big(\|\mathbf{u}-\mathbf{u}_h\|^2_{\mathbf{H}^1(\tau_l^1\cup\tau_l^2)} + \|p_h - p\|^2_{L^2(\tau_l^1\cup\tau_l^2)}\Big) \\
&\quad + \int_{\tau_l^1\cup\tau_l^2}\Big(\nu\Delta\mathbf{u}_h - p_h + f - (\mathbf{u}\cdot\nabla)\mathbf{u} + T\mathbf{g}\Big)\mathbf{W}_l + C\delta\|\mathbf{W}_l\|^2_{H^1(\tau_l^1\cup\tau_l^2)} \\
\le& C(\delta)\Big(\|\mathbf{u}-\mathbf{u}_h\|^2_{\mathbf{H}^1(\tau_l^1\cup\tau_l^2)} + \|T-T_h\|^2_{H^1(\tau_l^1\cup\tau_l^2)} + \|p_h - p\|^2_{L^2(\tau_l^1\cup\tau_l^2)}\Big) \\
&\quad + C\int_{\tau_l^1\cup\tau_l^2} h_\tau^2\Big|f + \nu\Delta\mathbf{u}_h - (\mathbf{u}_h\cdot\nabla)\mathbf{u}_h - \nabla p_h + T_h\mathbf{g}\Big|^2 \\
&\quad + C\delta\Big(h_\tau^{-2}\|\mathbf{W}_l\|^2_{L^2(\tau_l^1\cup\tau_l^2)} + \|\mathbf{W}_l\|^2_{H^1(\tau_l^1\cup\tau_l^2)}\Big).
\end{aligned}$$

Hence,

$$\begin{aligned}
&\sum_\tau \int_l h_l[\nu\nabla\mathbf{u}_h\cdot n - p_h\mathbf{n}]^2 \\
\le\ & C\Big(\|\mathbf{u}-\mathbf{u}_h\|^2_{\mathbf{H}^1(\Omega)} + \|T-T_h\|^2_{H^1(\Omega)} + \|p_h - p\|^2_{L^2(\Omega)}\Big) \\
&\quad + C\sum_\tau\int_\tau h_\tau^2\Big|f + \nu\Delta\mathbf{u}_h - (\mathbf{u}_h\cdot\nabla)\mathbf{u}_h - \nabla p_h + T_h\mathbf{g}\Big|^2. \qquad (4.36)
\end{aligned}$$

Therefore, this proves (4.29). □

Similarly, we can prove the following lower bound estimate for $\eta_8, ...\eta_{13}$.

Lemma 4.9. *Let* $(\boldsymbol{u}, T, p)$ *and* $(\boldsymbol{u}_h, T_h, p_h)$ *be the solutions of* (2.4) *and* (4.6) *respectively. Let* $(\boldsymbol{w}, \varphi, \sigma)$ *and* $(\boldsymbol{w}_h, \varphi_h, \sigma_h)$ *be the solutions of the co-state equations* (2.5) *and* (4.7) *respectively. Then,*

$$\begin{aligned}
\sum_8^{13}\eta_i^2 \le C\Big(&\|Q-Q_h\|^2_{0,\Omega} + \|\boldsymbol{u}-\boldsymbol{u}_h\|^2_{\boldsymbol{H}^1(\Omega)} + \|T-T_h\|^2_{H^1(\Omega)} + \|p-p_h\|^2_{0,\Omega} \\
&+\|\boldsymbol{w}-\boldsymbol{w}_h\|^2_{\boldsymbol{H}^1(\Omega)} + \|\varphi-\varphi_h\|^2_{H^1(\Omega)} + \|\sigma-\sigma_h\|^2_{0,\Omega}\Big) + C\epsilon_2^2 + C\epsilon_3^2, \qquad (4.37)
\end{aligned}$$

where η_i *is defined in Lemma* 4.6*,*

$$\epsilon_3^2 = \sum_{\tau\in T^h}\int_\tau h_\tau^2(\mathbf{U}-\bar{\mathbf{U}})^2.$$

Using Lemmas 4.8 and 4.9, we can have the following a posteriori lower bound.

Theorem 4.2. *Let* $(\boldsymbol{u}, T, p)$ *and* $(\boldsymbol{u}_h, T_h, p_h)$ *be the solutions of* (2.4) *and* (4.6) *respectively. Let* $(\boldsymbol{w}, \varphi, \sigma)$ *and* $(\boldsymbol{w}_h, \varphi_h, \sigma_h)$ *be the solutions of the co-state equations* (2.5) *and* (4.7) *respectively.*

Assume that all the conditions of above lemmas are also valid. For sufficient small h, then it gives

$$\eta_1^2+\sum_6^{13}\eta_i^2 \le C\Big(e^2+\|Q-Q_h\|_{0,\Omega}^2+\|\boldsymbol{u}-\boldsymbol{u}_h\|_{\boldsymbol{H}^1(\Omega)}^2+\|T-T_h\|_{H^1(\Omega)}^2+\|p-p_h\|_{0,\Omega}^2 + \|\boldsymbol{w}-\boldsymbol{w}_h\|_{\boldsymbol{H}^1(\Omega)}^2+\|\varphi-\varphi_h\|_{H^1(\Omega)}^2+\|\sigma-\sigma_h\|_{0,\Omega}^2\Big)+C\epsilon_2^2+C\epsilon_3^2, \tag{4.38}$$

where η_i,ϵ_i are defined before.

Proof. Now based on the above lemmas, we only need to estimate η_1. Note that $\alpha Q+\varphi=0$ when $Q>d$ and $\alpha d+\varphi\ge 0$ when $Q=d$. Let

$$\Omega_d^d=\{x\in\Omega_d^-:\ Q(x)=d\}.$$

We have that

$$\begin{aligned}\int_{\Omega_d^-}(\alpha Q_h+\varphi_h)^2=&\int_{\Omega_d^d}\Big(\alpha Q_h+\varphi_h-\alpha Q+\alpha d)\Big)^2+\int_{\Omega_d^-\backslash\Omega_d^d}\Big(\alpha Q_h+\varphi_h-\alpha Q-\varphi\Big)^2\\ \le&C\Big(\|Q-Q_h\|_{L^2(\Omega)}^2+\|\varphi-\varphi_h\|_{L^2(\Omega)}^2+\int_{\Omega_d^d}(\varphi_h+\alpha d)^2\Big)\\ \le&C\Big(\|Q-Q_h\|_{L^2(\Omega)}^2+\|\varphi-\varphi_h\|_{L^2(\Omega)}^2+\int_{\Omega_d^d}(\varphi_h+\alpha d-\varphi-\alpha d)^2\Big)\\ \le&C\Big(\|Q-Q_h\|_{L^2(\Omega)}^2+\|\varphi-\varphi_h\|_{L^2(\Omega)}^2\Big),\end{aligned}$$

where we used the facts that $\varphi_h+\alpha d\le 0\le\varphi+\alpha d$ on Ω_d^d. Moreover, note that $Q>d$ and hence $\alpha Q+\varphi=0$ on $\Omega_d^+\setminus\Omega_*$. It can be deduced that

$$\begin{aligned}&\int_{\Omega_d^+}(\alpha Q_h+\varphi_h)^2\\ =&\int_{\Omega_*}(\alpha Q_h+\varphi_h)^2+\int_{\Omega_d^+\backslash\Omega_*}(\alpha Q_h+\varphi_h)^2\\ =&\int_{\Omega_*}\Big(\alpha Q_h+\varphi_h-\mathcal{R}_h(\alpha Q_h+\varphi_h)\Big)^2+\int_{\Omega_d^+\backslash\Omega_*}\Big(\alpha Q_h+\varphi_h-(\alpha Q+\varphi)\Big)^2\\ \le& C\int_{\Omega_*}\Big(\alpha Q+\varphi-\mathcal{R}_h(\alpha Q+\varphi)\Big)^2+C\int_{\Omega_*}\Big(\alpha Q_h+\varphi_h-(\alpha Q+\varphi)\Big)^2\\ &+C\int_{\Omega_*}\Big(\mathcal{R}_h(\alpha Q+\varphi)-\mathcal{R}_h(\alpha Q_h+\varphi_h)\Big)^2+C\big(\|Q-Q_h\|_{0,\Omega}^2+\|\varphi-\varphi_h\|_{0,\Omega}^2\big)\\ \le& Ce^2+C\big(\|Q-Q_h\|_{0,\Omega}^2+\|\varphi-\varphi_h\|_{0,\Omega}^2\big).\end{aligned} \tag{4.39}$$

So, we complete our proof. □

5. Concluding Remarks

In this paper we develop the adaptive finite element approximation for the distributed optimal control associated with the stationary Bénard problem under the pointwise control constraint. We give the a posteriori error estimates mainly with the H^1-norm appearances for the states and co-states and for the control with the L^2-norm appearance. In the further research, the a posteriori error estimates for the states and co-states with the L^2-norm appearances will be considered, and especially the a posteriori lower bound for $\eta_2, \cdots, \eta_5$ and $\eta_4, \cdots, \eta_{17}$ will be given using the new bubble functions.

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