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ON THE DISCRETE MAXIMUM PRINCIPLE FOR THE LOCAL PROJECTION SCHEME WITH SHOCK CAPTURING*

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Abstract

It is a well known fact that finite element solutions of convection dominated problems can exhibit spurious oscillations in the vicinity of boundary layers. One way to overcome this numerical instability is to use schemes that satisfy the discrete maximum principle. There are monotone methods for piecewise linear elements on simplices based on the upwind techniques or artificial diffusion. In order to satisfy the discrete maximum principle for the local projection scheme, we add an edge oriented shock capturing term to the bilinear form. The analysis of the proposed stabilisation method is complemented with numerical examples in $2D$.

Mathematics subject classification: 65N30, 65M60
Key words: Local projection stabilization, Discrete maximum principle, Shock capturing

1. Introduction

Convection-diffusion-reaction equations occur for instance in chemical engineering. Depending on the problem, different types of boundary conditions are applied on different parts of the domain boundary. A common feature of these problems is the small diffusion coefficient, i.e., the process is convection and/or reaction dominant. Since standard Galerkin discretisations will produce unphysical oscillations for this type of problems, stabilisation techniques have been developed. The streamline-upwind Petrov–Galerkin method (SUPG) has been successfully applied to convection-diffusion-reaction problems. It was proposed by Hughes and Brooks [19]. One fundamental drawback of SUPG is that several terms which include second order derivatives have to be added to the standard Galerkin discretisation in order to ensure consistency. Alternatively, continuous interior penalty methods [1,5], residual free bubble method [13–15], or subgrid modelling [11,17] can be used for stabilising the discretised convection-diffusion-reaction problems.

Despite of well investigated stabilising effects of the local projection method (LPS) for scalar convection-diffusion problems and its relations to other stabilisation methods like SUPG and continuous interior penalty methods (CIP), see [26, 29, 30], the problem of spurious oscillations at the boundary layer arises. This lack of numerical stability can lead to solutions which do not preserve physical properties, e.g., non-negativity of concentration of chemical species. Expressing this issue mathematically, we can say that numerical solutions do not satisfy the maximum principle in certain sense. The pioneering work on the field of discrete maximum principle for finite elements is [8]. The authors showed that the solution with continuous

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piecewise linear shape functions satisfies the discrete maximum principle for Poisson equation on weakly acute triangular meshes. Since then many improvements and extensions have been done. We mention [4, 10, 18, 20, 23–25, 27, 31–35] as the most important results of the last decades. Undesired spurious oscillations can be also reduced or even eliminated by employing a suitable choice of stabilising parameters in order to get a nodally exact solutions, see [30]. Another possibility of satisfying the discrete maximum principle is the use of additional terms, see review article [21] and [22] for the detailed discussion of the optimal choice of stabilising parameters. It has been proved that the first-order artificial viscosity scheme of [9] and nonlinear artificial scheme of [6] produce solutions which satisfy the discrete maximum principle. The rigorous proof for discrete maximum principle for CIP scheme perturbated by shock capturing term has been established in [7]. Recently, the discrete maximum principle has been proved for finite element solutions of convection-diffusion equations by using edge-based nonlinear diffusion, cf., [3], whereas it is also proved for transport problems with non-smooth data by using symmetric projection stabilization technique, cf., [2].

Our aim is to establish a discrete maximum principle of the local projection scheme modified by shock capturing (LPSSC) and to provide the convergence theory of this method applied to convection-diffusion-reaction problems with Dirichlet boundary conditions. Furthermore, several test problems with different types of interior and boundary layers will be presented. Our numerical tests show that the local projection stabilisation with the edge oriented shock capturing allows to obtain numerical solutions which are physically reliable. However, in the case of smooth solutions we have suboptimal order of convergence.

2. Model Problem and Local Projection Method with Shock Capturing

2.1. Weak formulation

We consider the following Dirichlet problem for the scalar convection-diffusion-reaction equation in two dimensions

$$\begin{cases} -D\Delta u + \boldsymbol{b}\cdot\nabla u + cu = f & \text{in } \Omega, \\ u = g & \text{on } \Gamma = \partial\Omega\,, \end{cases} \tag{2.1}$$

where $D > 0$ is a small diffusion constant. We are looking for the distribution of concentration or temperature u in a polygonally bounded reactor domain $\Omega \subset \mathbb{R}^2$. The reaction coefficient $c \in L^\infty(\Omega)$ is assumed to be non-negative. Let $f \in L^2(\Omega)$, $g \in H^{1/2}(\Gamma)$ be given functions. Furthermore, we require that the convection field $\boldsymbol{b} \in \left(W^{1,\infty}(\Omega)\right)^n$, $n = 2$, and the reaction coefficient c fulfils for some $c_0 > 0$ the following condition

$$c(x) - \frac{1}{2}\nabla\cdot\boldsymbol{b}(x) \geq c_0 > 0 \qquad \forall x \in \overline{\Omega}. \tag{2.2}$$

We define the function spaces

$$V = H^1(\Omega) \quad \text{and} \quad V_0 = \{v \in V:\ v|_\Gamma = 0\}.$$

A weak formulation of (2.1) reads: Find $u \in V$ with $u|_\Gamma = g$ such that

$$a(u,v) = (f,v) \quad \forall\, v \in V_0 \tag{2.3}$$