Journal of Computational Mathematics
Vol.35, No.5, 2017, 600–619.

http://www.global-sci.org/jcm
doi:10.4208/jcm.1607-m2016-0613

ON PMHSS ITERATION METHODS FOR CONTINUOUS SYLVESTER EQUATIONS*

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Abstract

The modified Hermitian and skew-Hermitian splitting (MHSS) iteration method and preconditioned MHSS (PMHSS) iteration method were introduced respectively. In the paper, on the basis of the MHSS iteration method, we present a PMHSS iteration method for solving large sparse continuous Sylvester equations with non-Hermitian and complex symmetric positive definite/semi-definite matrices. Under suitable conditions, we prove the convergence of the PMHSS iteration method and discuss the spectral properties of the preconditioned matrix. Moreover, to reduce the computing cost, we establish an inexact variant of the PMHSS iteration method and analyze its convergence property in detail. Numerical results show that the PMHSS iteration method and its inexact variant are efficient and robust solvers for this class of continuous Sylvester equations.

Mathematics subject classification: 65F10, 65F50.
Key words: Continuous Sylvester equation, PMHSS iteration, Inexact PMHSS iteration, Preconditioning, Convergence.

1. Introduction

For solving a class of complex symmetric linear systems $(W + iT)x = b$, where $i = \sqrt{-1}$, $W, T \in \mathbb{R}^{n\times n}$ are real, symmetric, and positive semidefinite matrices with at least one of them, being positive definite, Bai et al. introduced the MHSS iteration method [1] and the PMHSS iteration method [2, 3], respectively. Moreover, for solving large sparse continuous Sylvester equations with non-Hermitian and positive definite/semidefinite matrices, Bai presented a Hermitian and skew-Hermitian splitting (HSS) iteration method [4, 5]. For more details about the HSS iteration method and theory, we refer to [5–11] and the references therein. According to the HSS iteration method, Zhou et al. proposed a MHSS iteration method for solving large sparse continuous Sylvester equations with non-Hermitian and complex symmetric positive definite/semidefinite matrices [12]. Recently, a MHSS iteration method was also presented for solving the complex linear matrix equation $AXB = C$ [13, 14].

In the paper, we consider the iteration solution of the following continuous Sylvester equation

$$AX + XB = F, \tag{1.1}$$

where $A \in \mathbb{C}^{m\times m}$, $B \in \mathbb{C}^{n\times n}$ and $F \in \mathbb{C}^{m\times n}$ are given complex matrices. Assume that A, B and F are large and sparse matrices. Let $A = W + iT$ and $B = U + iV$, where $W, T \in \mathbb{R}^{m\times m}$, $U, V \in \mathbb{R}^{n\times n}$ are real symmetric matrices, with W being positive definite and T, U, V positive semi-definite. We assume $T \neq 0$, which implies that A is non-Hermitian. The continuous

* Received April 21, 2016 / Revised version received July 12, 2016 / Accepted July 26, 2016 / Published online July 1, 2017 /

Sylvester equation with $A = W + iT$ and $B = U + iV$ may arise from numerical solutions of PDEs with complex coefficients [1, 2]. A Lyapunov equation is a special case of the Sylvester equations, where $B = A^*$ and $F = F^*$. It is well known that the continuous Sylvester equation (1.1) has a unique solution, under the assumes that there is no common eigenvalue between A and $-B$ [4]. The continuous Sylvester equation (1.1) is mathematically equivalent to the system of linear equations

$$\mathcal{A}x = f, \tag{1.2}$$

where $\mathcal{A} = I \otimes A + B^T \otimes I$, and the vector x and f contain the concatenated columns of the matrices X and F, respectively, with $\otimes$ being the Kronecker product symbol and B^T representing the transpose of the matrix B.

The matrix equation (1.1) plays an important role in numerical methods for differential equations with complex coefficients [1,2,4], iterative methods for algebraic Riccati equations [15–17], matrix nearness problem [18], image restoration [19] and other problems; see [4, 12, 20–31] and the references therein. Recent interest is directed more towards large and sparse matrices A and B, and $F = CD^*$ with very low rank, where C and D have only a few columns [32]. In these cases, the standard methods are often too expensive to be practical, and iterative methods become more viable choices [14, 25]. The standard direct method for solving (1.1) is due to Bartels and Stewart [26]. However, this method requires dense matrix operations such as the Schur decomposition; thus is not applicable in large-scale settings. For large-scale settings, iterative methods have been developed that take advantage of the sparsity and the low-rank structure. The two most common ones are the Alternating Direction Implicit (ADI) method [25, 30, 32] and the (rational) Krylov projection methods [28]. Advantages of Krylov subspace based algorithms over ADI iterations are that no knowledge about the spectra of A and B is needed and (except for [27]) no linear systems of equations with (shifted) A and B have to be solved. But ADI iterations often enable faster convergence if (sub) optimal shifts to A and B can be effectively estimated [25]. Recently, Ding and Chen proposed a few simple iterative schemes, namely, Gradient based iterative (GI) algorithms, for matrix equations [29] (and others therein). The schemes, resembling the classical Jacobi and Gaussian iterations for linear systems, are easy to implement and cost little per step but converge linearly at the best [25, 29, 33, 34].

In the paper, we mainly conduct the idea of the HSS based iteration method to solve the continuous Sylvester equation (1.1), see, e.g., [1–14,17,19–23,33–35] and references therein. Bai, Golub and Ng in [5] firstly proposed the Hermitian and skew-Hermitian splitting (HSS) method for non-Hermitian positive-definite linear systems. Because of the effectiveness and robustness of the HSS method, it is extensively studied and extended to other equations and conditions, see e.g. [6–9] and references therein. A considerable advantage of the MHSS iteration [1, 12] consists in the fact that the solutions of the shifted skew-Hermitian sub-system of the continuous Sylvester equation with coefficient matrices $\alpha I + iT$ and $\beta I + iV$ are avoided and only two linear sub-systems with real and symmetric positive definite coefficient matrices need to be solved at each step [1,12]. Therefore, operations on these matrices can be carried out using real arithmetic only. To the best of our knowledge, there are not a preconditioned MHSS iteration to solve the continuous Sylvester equation (1.1) [2, 9, 36, 37]. Motivated by this, we further propose and analyze a new iteration approach called PMHSS for solving the continuous Sylvester equation (1.1).

The rest of this paper is organized as follows. In Section 2, after a brief introduction of the Smith method [17, 38], the MHSS iteration method [1, 12, 13], we present a PMHSS iteration