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Norm Inequalities for Fractional Integral Operators on Generalized Weighted Morrey Spaces

Yueshan Wang*

Department of Mathematics, Jiaozuo University, Jiaozuo, Henan 454003, China

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Abstract. Considering a class of operators which include fractional integrals related to operators with Gaussian kernel bounds, the fractional integral operators with rough kernels and fractional maximal operators with rough kernels as special cases, we prove that if these operators are bounded on weighted Lebesgue spaces and satisfy some local pointwise control, then these operators and the commutators of these operators with a BMO functions are also bounded on generalized weighted Morrey spaces.

Key Words: Fractional integral, rough kernel, Gaussian kernel bound, commutator, generalized weighted Morrey space.

AMS Subject Classifications: 42B20, 47G10, 42B35

1 Introduction

The classical Morrey spaces were introduced by Morrey [1] in 1938, since then a large number of investigations have been given to them by mathematicians. It is well-known that the classical Morrey spaces and the weighted Lebesgue spaces play important roles in the theory of partial differential equations.

Let $1\le p<\infty$ and $0\le\lambda\le n$, Morrey spaces are defined by

$$L^{p,\lambda}(\mathbb{R}^n)=\Big\{f\in L^p_{loc}(\mathbb{R}^n):\|f\|_{L^{p,\lambda}(\mathbb{R}^n)}<\infty\Big\},$$

where

$$\|f\|_{L^{p,\lambda}(\mathbb{R}^n)}=\sup_{x\in\mathbb{R}^n,r>0} r^{-\lambda/p}\|f\|_{L^p(B(x,r))}. \tag{1.1}$$

Note that $L^{p,0}(\mathbb{R}^n)=L^p(\mathbb{R}^n)$ and $L^{p,n}(\mathbb{R}^n)=L^\infty(\mathbb{R}^n)$. If $\lambda<0$ or $\lambda>n$, then $L^{p,\lambda}(\mathbb{R}^n)=\Theta$, where Θ is the set of all functions equivalent to 0 on $\mathbb{R}^n$.

*Corresponding author. *Email address:* `wangys1962@163.com` (Y. S. Wang)

Let $\Phi(r)$, $r>0$ be a growth function, that is, a positive increasing function in $(0,\infty)$ and satisfies doubling condition

$$\Phi(2r)\le D\Phi(r) \quad \text{for all } \ r>0,$$

where $D=D(\Phi)\ge 1$ is a doubling constant independent of r. In [2], Mizuhara gave a generalization Morrey spaces $L^{p,\Phi}(\mathbb{R}^n)$ considering $\Phi(r)$ instead of r^λ in (1.1). He studied also a continuity in $L^{p,\Phi}(\mathbb{R}^n)$ of some classical integral operators.

Komori and Shirai [3] introduced a version of the weighted Morrey space $L^{p,\kappa}(\omega,\mathbb{R}^n)$, which is a natural generalization of the weighted Lebesgue space $L^p(\omega,\mathbb{R}^n)$. Let $1\le p<\infty$, $0<\kappa<1$ and ω be a weight function. The spaces $L^{p,\kappa}(\omega,\mathbb{R}^n)$ are defined by

$$L^{p,\kappa}(\omega,\mathbb{R}^n)=\{f\in L^p_{loc}(\omega):\|f\|_{L^{p,\kappa}(\omega,\mathbb{R}^n)}<\infty\},$$

where

$$\|f\|_{L^{p,\kappa}(\omega,\mathbb{R}^n)}=\sup_{x\in\mathbb{R}^n,r>0}\left(\frac{1}{\omega(B(x,r))^\kappa}\int_{B(x,r)}|f(y)|^p\omega(y)dy\right)^{\frac{1}{p}}.$$

In order to deal with the fractional order case, we need to consider the weighted Morrey space with two weights, it was also introduced by Komori and Shirai in [3].

Let $1\le p<\infty$, $0<\kappa<1$. Then for two weights u, v, the weighted Morrey space is defined by

$$L^{p,\kappa}(u,v)(\mathbb{R}^n)=\{f\in L^p_{loc}(u):\|f\|_{L^{p,\kappa}(u,v)}<\infty\},$$

where

$$\|f\|_{L^{p,\kappa}(u,v)(\mathbb{R}^n)}=\sup_{x\in\mathbb{R}^n,r>0}\left(\frac{1}{v(B(x,r))^\kappa}\int_{B(x,r)}|f(y)|^p u(x)dy\right)^{\frac{1}{p}}.$$

Let $1\le p<\infty$, φ be a positive measurable function on $\mathbb{R}^n\times(0,\infty)$ and ω be a non-negative measurable function on $\mathbb{R}^n$. We denote by $M^p_\varphi(\omega,\mathbb{R}^n)$ the generalized weighted Morrey space, the space of all functions $f\in L^p_{loc}(\omega,\mathbb{R}^n)$ with finite norm

$$\|f\|_{M^p_\varphi(w,\mathbb{R}^n)}=\sup_{x\in\mathbb{R}^n,r>0}\frac{1}{\varphi(x,r)}\left(\frac{1}{w(B(x,r))}\|f\|^p_{L^p(\omega,B(x,r))}\right)^{1/p},$$

If $\omega=1$ and $\varphi(x,r)=r^{\frac{\lambda-n}{p}}$ with $0\le\lambda\le n$, then $M^p_\varphi(\omega,\mathbb{R}^n)=L^{p,\lambda}(\mathbb{R}^n)$ is the classical Morrey space; If $\varphi(x,r)=\omega(B(x,r))^{\frac{\kappa-1}{p}}$, $0<\kappa<1$, then $M^p_\varphi(\omega,\mathbb{R}^n)=L^{p,\kappa}(\omega,\mathbb{R}^n)$ is the weighted Morrey space; If $\omega=1$, $\varphi(x,r)^p=\Phi(r)r^{-n}$, then $M^p_\varphi(\omega,\mathbb{R}^n)=L^{p,\Phi}(\mathbb{R}^n)$.

It has been proved by many authors (see [4–8]) that most of the operators which are bounded on a weighted (unweighted) Lebesgue space are also bounded in an appropriate weighted (unweighted) Morrey space. In this paper, we are going to prove that these results are valid on generalized weighted Morrey space. Our main results can be formulated as follows.