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A NEW FINITE ELEMENT FOR INTERFACE PROBLEMS HAVING ROBIN TYPE JUMP

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Abstract. We propose a new finite element method for solving second order elliptic interface problems whose solution has a Robin type jump along the interface. We cast the problem into a new variational form and introduce a finite element method to solve it using a uniform grid. We modify the P_1-Crouzeix-Raviart element so that the shape functions satisfy the jump conditions along the interface. We note that there are cases that the Lagrange type basis can not be used because of the jump in the value. Numerical experiments are provided.

Key words. Finite element, uniform grid, interface problem, Robin type jump, P_1-Crouzeix-Raviart finite element method.

1. Introduction

In recent years, there has been an extensive research towards problems involving interface, (see [1, 15, 22, 37, 35, 39, 41] and references therein) and numerical methods for such problems. A widely studied example is an elliptic problem having discontinuous coefficients, where the solution satisfies natural jump conditions $[u] = 0$, $[\beta \frac{\partial u}{\partial n}] = 0$ across the interface immersed in domain. See [14, 21, 23, 40, 41], for example. This kind of problem typically arises from diffusion phenomena in a material consisting of heterogeneous media. Other important class of problem includes the time-dependent problems which may have a moving interface [24, 33, 38, 42], for instance, the incompressible Navier-Stokes equations for two fluids [19, 36] and an solid/solid or solid/fluid interaction problems [8, 9, 22]. In most of those examples, the primary variables, such as heat, potential, displacement and velocity, etc., or their derivatives (or flux) have certain jumps. To solve such problems numerically, for instance by finite element method, one usually need to use body fitted grids to get the optimal numerical results. But the grid generation is complicated and it is a time consuming job to solve the linear equation derived from the body fitted grids since the matrix is unstructured.

On the other hand, a new class of finite element methods have been suggested and are shown to perform quite well for interface problems, see [14, 30, 41] and references therein. These methods are called *immersed finite element methods (IFEMs)* which use non fitted (say, uniform) grid for interface problems (so the interface cuts the interior of some elements). The idea of this new method is to modify the basis functions so that they satisfy the interface conditions along the interface within each element. Although the first one of these schemes was proposed for the finite element methods using Lagrange type P_1 element having degree of freedom at vertices, the idea works well especially with Crouzeix-Raviart(CR) nonconforming basis functions [17], since the consistency error term can be shown to be optimal when CR bases are used [30].

Let us briefly review some works related to general interface conditions. Angot [46] proposed a fictitious domain method to embed a smooth domain into a

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rectangular domain and showed that the two formulations are equivalent. In the meantime, the boundary condition was transformed into an interface condition. The numerical scheme using uniform grids was proposed in [4]. They used standard piecewise linear basis function together with the local refinement to resolve the smooth interface. Similar approach using finite volume methods was earlier suggested in [3]. For the problem with natural interface conditions, Ji et al. [25] have studied similar problems using the standard basis function on each sub element, hence they need extra basis functions. Their scheme also has the problem of severe deterioration of condition number. Hence they proposed adding a ghost-penalty suggested in [12] to stabilize the condition number of the resulting matrix. There are other types of unfitted grid method, see [21], [22] and references therein, where one uses cut basis functions as extra degree of freedoms. The case of elasticity problems with homogeneous condition using rectangular grids was considered in [44] and the analysis for triangular grids is carried out in [31]. One dimensional poroelasticity problem using IIM was considered by Bean et al. [7]. Furthermore, coupled Darcy flow and Stokes flow are studied in [34] and the numerical method based on DG scheme has appeared in [47].

In this paper, we propose a new IFEM scheme using CR nonconforming basis functions which can handle jump discontinuity of different kinds. All of the schemes discussed above have certain similarity with our scheme in the sense that they all use unfitted grids. However, they are different from ours at least one of the following aspect; either (i) they treat homogenous jump condition only (the solution is thus continuous), or (ii) they use Lagrange type P_1 nodal basis functions, or (iii) they do not consider consistency terms to compensate the errors, or (iv) they use extra degrees of freedom to capture the discontinuity along the interface, or (v) their scheme have the problem of ill-conditioning for certain interface. Our scheme to be presented here does not have any disadvantages/restrictions above.

Now we describe the model equation with interface, where the jump of primary variable is related to the normal flux. These problems arise in the study of medical imaging of cancer cells such as MREIT [1, 2], problems with spring-type jumps in structural mechanics [29, 45], or electrochemotherapy [33], where the conductivity of cell membrane changes abruptly across the membrane. In the development of MREIT, for example, we encounter a partial differential equation (PDE) which models the electric behavior of biological tissue under the influence of an electric field which involves many cells. Conducting cytoplasm is surrounded by a thin insulating membrane (see Figure 1). Inside each cell Ω^i, $i = 1, \dots, N$, the medium is homogeneous and isotropic. We assume that the conductivity of the cell Ω^i is β for all i. The outside of cells, which is denoted by Ω^0, is also composed of an isotropic homogeneous medium whose conductivity is β. Let $\Omega := \cup_{i=0}^{N} \Omega^i$ be a whole domain. The membrane of the cell is very thin and resistive. The thickness d of the membrane is very small compared to the size of the cells, i.e., $d \ll |\Omega^i|$. Since the membrane is very resistive, the conductivity of the membrane β_{mem} is close to zero. The derived model equation depends on the value of conductivity. Similar description may apply to electrochemotherapy. In such problems, the electric potential or