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# A Posteriori Error Estimation of Spectral and Spectral Element Methods for the Stokes/Darcy Coupled Problem

Weiwei Wang, Chuanju Xu*

*School of Mathematical Sciences, Xiamen University, 361005 Xiamen, China*

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**Abstract.** In this paper, we carry out an *a posteriori* error analysis of Legendre spectral approximations to the Stokes/Darcy coupled equations. The spectral approximations are based on a weak formulation of the coupled equations by using the Beavers-Joseph-Saffman interface condition. The main contribution of the paper consists of deriving a number of posteriori error indicators and their upper and lower bounds for the single domain case. An extension of the upper bounds to the multi-domain case in the spectral element framework is also given.

**AMS subject classifications**: 35Q35, 76D03

**Key words**: *A posteriori* error, Stokes/Darcy coupled equations, Spectral method, Spectral element method.

## 1 Introduction

The model of the Stokes equations coupled with the Darcy equations has been a subject of interest in a large variety of different fields, see, e.g. [9–12,14,16]. Recently, we have introduced a new formulation for the Stokes/Darcy coupled equations, subject respectively to the Beavers-Joseph-Saffman interface condition and an alternative matching interface condition [22]. Some spectral approximations are proposed and a priori error estimates are derived therein. In this paper we consider *a posteriori* error analysis for the above mentioned spectral approximations. The motivation of this consideration is that *a posteriori* error estimators are computable quantities in terms of the discrete solution, and can be used to measure the actual approximation errors without the knowledge of exact solutions. They are essential for designing algorithms with adaptive mesh refinement with minimal computational cost. On the other side, there are few work on *a posteriori* error analysis of the spectral method, and it is not clear if the adaptive strategy in the spectral method can be as efficient as in the finite element framework. Therefore this paper can be

*Corresponding author. *Email addresses:* cjxu@xmu.edu.cn (C.J. Xu)

 

regarded as a step towards a better understanding about the adaptive spectral method, with particular attention to the Stokes/Darcy coupled equations.

Some *a posteriori* error analysis of the finite element approximation to the Stokes Darcy coupled equations have been carried in [1, 8]. The work [1] used the Lagrangian multiplier in their variational formulation, while [8] replaced the Darcy equations with a Poisson-like equation. In [1] the finite element subspaces consist of Bernardi-Raugel and Raviart-Thomas elements for the velocities, piecewise constants for the pressures, and continuous piecewise linear elements for the Lagrange multiplier defined on the interface. They have derived a residual-based *a posteriori* error estimate for the Stokes/Darcy coupled problem. The finite element spaces adopted in [8] are the Hood-Taylor element for the velocity and the pressure in the Stokes equation and conforming piecewise quadratic element for the Darcy pressure. The *a posteriori* error analysis was based on a suitable evaluation of the residual of the finite element solution.

In contrast to the lower order methods, *a posteriori* error estimation for high order methods such as spectral method is much less developed, although there exist a few papers on this topic for the elliptic problems (see, e.g., [3, 6, 13, 19]). The purpose of this work is to carry out an *a posteriori* error analysis for the spectral approximation of the Stokes/Darcy coupled equations. The analysis will be based on the formulation introduced in our previous work, which allows to extend the idea from [1,6,8,19] to derive the residual-based a posteriori error estimator in the framework of spectral element method.

The rest of the paper is organized as follows. In section 2 we briefly recall the formulation proposed in [22] for the Stokes/Darcy coupled problem. The core of the work is given by section 3 and section 4, where we develop the *a posteriori* error analysis. In section 3 we derive a residual-based *a posteriori* error estimate. The efficiency of this estimate is given in section 4. In section 5, we extend the results to the case of multi-domain in the framework of the spectral element method.

Throughout the paper we use the standard terminology for Sobolev spaces. In particular, If $D$ is a bounded connected domain and $r\in\mathbb{R}$, then $|\cdot|_{r,D}$ and $\|\cdot\|_{r,D}$ stand for the semi-norm and norm in the Sobolev spaces $H^r(D)$, $[H^r(D)]^2$, and $[H^r(D)]^{2\times 2}$. In what follows, we will use $c$ to mean a generic positive constant independent of any functions and of any discretization parameters. We also use the expression $A\lesssim B$ to mean that $A\leqslant cB$.

## 2 The Stokes/Darcy coupled problem

We are interested in the following Stokes/Darcy coupled equations in two dimensions: