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A Survey of the Matrix Exponential Formulae with Some Applications

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Abstract. The matrix exponential formulae is a very important tool in diverse fields of mathematical physics, in particular, it is also useful in studying quantum information theory. In this paper, we survey results related to matrix exponential thoroughly. The purpose of the article is pedagogical.

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Key words: Matrix, exponential formula, quantum information.

1 Introduction and preliminary

The matrix exponential formulae have applications in diverse fields in mathematical physics. For example, one can apply it to prove the famous strong subadditivity of the von Neumann entropy, and the Bessis-Moussa-Villani conjecture [16]. We use these results to study also some interesting quantum informational problems [24–29]. In this paper, we present a detailed survey, where some emphasis is put on the Lie-Trotter-Suzuki product formulae, Thompson formula, Wasin-So formula, Stahl's theorem, Peierls- Bogoliubov inequality, Golden-Thompson inequality, reverse inequality to Golden-Thompson type inequalities, trace inequality in quantum information theory, Itzykson-Zuber integral formula, etc. We just collect similar results from various places and put together. Although there are no new results in this article, but some detailed and elementary proofs of partial results are still included for completeness. Firstly, we state some fundamental knowledge of matrix exponential map, which is based on [11].

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1.1 One-parameter groups of linear transformations

In this section, we show how one-parameter groups of linear transformation of a vector space can be described using the exponential map on matrices. Let V be a finite dimensional vector space, $\mathrm{End}(V)$ denote the algebra of linear maps from V to itself, and $\mathrm{GL}(V)$ denote the group of invertible linear maps from V to itself. The usual name for $\mathrm{GL}(V)$ is the *general linear group* of V. If $V=\mathbb{K}^n$, then $\mathrm{End}(V)=M_n(\mathbb{K})$, the $n\times n$ matrices over field $\mathbb{K}$, where $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$, and $\mathrm{GL}(V)=\mathrm{GL}_n(\mathbb{K})$, the matrices with non-vanishing determinants. Let $(V,\|*\|)$ be a normed space, where the norm is defined by

$$\|A\|\stackrel{\text{def}}{=}\sup\left\{\frac{\|Av\|}{\|v\|}:0\neq v\in V\right\},\quad A\in\mathrm{End}(V). \tag{1.1}$$

Definition 1.1. A *one-parameter group of linear transformations* of V is a continuous homomorphism

$$M:\mathbb{R}\longrightarrow\mathrm{GL}(V).$$

That is, $M(t)$ is a collection of linear maps such that

(i) $M(0)=\mathbb{1}_V$,

(ii) $M(s)M(t)=M(s+t)\quad\forall s,t\in\mathbb{R}$,

(iii) $M(t)$ depends continuously on t.

Remark 1.1. (a) For $A\in\mathrm{End}(V)$ and $r>0$, set

$$B_r(A)\stackrel{\text{def}}{=}\Big\{X\in\mathrm{End}(V):\|X-A\|<r\Big\}.$$

The Neumann formula:

$$(\mathbb{1}_V-A)^{-1}=\sum_{n=0}^{\infty}A^n \tag{1.2}$$

valid for $A\in B_1(0)$ shows that

$$(B_r(\mathbb{1}_V))^{-1}\subseteq B_\alpha(\mathbb{1}_V) \tag{1.3}$$

with $\alpha=\frac{r}{1-r}$. Similarly, the formula

$$(\mathbb{1}_V+A)(\mathbb{1}_V+B)=\mathbb{1}_V+A+B+AB \tag{1.4}$$

shows

$$B_r(\mathbb{1}_V)B_s(\mathbb{1}_V)\subseteq B_{r+s+rs}(\mathbb{1}_V).$$

(b) For $A\in\mathrm{L}(V)$, define

$$\exp(A)\stackrel{\text{def}}{=}\sum_{n=0}^{\infty}\frac{A^n}{n!}. \tag{1.5}$$