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OPTIMAL COARSE GRID SIZE IN DOMAIN DECOMPOSITION*[1)]

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Abstract

In most domain decomposition (DD) methods, a coarse grid solve is employed to provide the global coupling required to produce an *optimal* method. The total cost of a method can depend sensitively on the choice of the coarse grid size H. In this paper, we give a simple analysis of this phenomenon for a model elliptic problem and a variant of Smith's vertex space domain decomposition method [11, 3]. We derive the optimal value H_{opt} which asymptotically minimizes the total cost of method (number of floating point operations in the sequential case and execution time in the parallel case), for subdomain solvers with different complexities. Using the value of H_{opt}, we derive the overall complexity of the DD method, which can be significantly lower than that of the subdomain solver.

1. Introduction

Domain decomposition (DD) is a class of techniques for solving elliptic boundary value problems in which the solution is obtained by iteratively solving smaller subdomain problems. These methods have received a lot of study in recent years (see [6, 1, 2, 7, 10]). They are attractive because of their inherent parallelism and their *optimal* convergence rates (i.e. independent of the mesh size). The optimality of the convergence rate requires the solution of a coarse grid problem at each iteration. The study of how to incorporate such a coarse grid solve in a DD method has received a lot of study in the literature.

The focus of our paper, however, is on the choice of the *size* H of the coarse grid. It is intuitively obvious that the total cost of a DD method can depend sensitively on this choice, in addition to the obvious dependence on the efficiency of the subdomain solver. A small H generally improves the convergence rate (because the coarse grid problem is a better approximation to the original fine grid problem) at the cost of a more costly coarse grid solve, whereas a large H has the opposite effect. Therefore,

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an optimal value H_{opt} often exists and indeed has been observed empirically [9, 12]. Surprisingly, there has been almost no systematic study in the literature on this issue. Our approach is to take a simple model elliptic problem and a particular DD method; this allows a simple but complete and easily understood analysis which we think give insights for more general situations.

For concreteness, we focus our analysis on a variant of Smith's vertex space method [11] developed by us earlier [3]. We consider subdomain solvers with different complexity, including banded Gaussian elimination, nested dissection, modified incomplete Cholesky factorization (MIC) and multigrid solvers. For simplicity, we assume the same solver is used for the subdomains and the coarse grid problem, and that these are solved exactly. By expressing the computational complexity as a function of the coarse grid size H and the fine grid size h, we derive the optimal value H_{opt} which *asymptotically* (as h tends to 0) minimizes the total cost of method. Using the value of H_{opt}, we can derive the overall complexity of the DD method as a function of h alone, which can be significantly lower than that of the subdomain solver. That is, through the use of DD, a given solver can be made more efficient for solving the original problem, *by using it to solve smaller (but more) sub-problems.* This is a simple consequence of the divide-and-conquer principle. The assumption of the asymptotic limit is not necessary but does allow a close form expression for H_{opt} from which one can see more clearly the general trend.

2. Fourier vertex space domain decomposition method

We consider the following 2nd order self adjoint elliptic problem on $\Omega \subset R^2$:

$$-\nabla \cdot (a(x,y)\nabla u) = f \text{ in } \Omega, \qquad u = 0 \text{ on } \partial\Omega, \tag{1}$$

with $a(x,y) \in R^{2\times 2}$ uniformly positive definite, bounded and piecewise smooth on Ω. We assume that the domain Ω is partitioned into N non-overlapping sub-domains $\Omega_1, \cdots, \Omega_N$ of diameter H,which form the elements of a *quasi-uniform* coarse grid triangulation τ^H. We also assume that the sub-domains Ω_i are refined to produce a fine grid *quasi-uniform* triangulation τ^h having elements of diameter h. Corresponding to the coarse grid and fine grid triangulations, we discretize (1) either by using finite elements, see [5], or by using finite difference methods, see [13], resulting in the symmetric positive definite linear systems $A_h u_h = f_h$ on the fine grid and $A_H u_H = f_H$ on the coarse grid.

Let I denote the union of the interiors of the sub-domains, and let B denote the interface separating the sub-domains, i.e. $I = \cup_i \Omega_i, B \equiv (\cup_i \partial\Omega_i) - \partial\Omega$. According to this partitioning, A_h, x and f can be written in block form:

$$A_h = \begin{pmatrix} A_{II} & A_{IB} \\ A_{BI} & A_{BB} \end{pmatrix} \qquad x = \begin{pmatrix} x_I \\ x_B \end{pmatrix} \qquad f = \begin{pmatrix} f_I \\ f_B \end{pmatrix}.$$