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ON THE UNIQUE SOLVABILITY OF THE NONLINEAR SYSTEMS IN MDBM*

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Abstract

To obtain the approximate solution of the nonlinear ordinary differential equations requires the solution to systems of nonlinear equations. The authors study the conditions for the existence and uniqueness of the solutions to the algebraic equations in multiderivative block methods.

1. Introduction

Consider the following initial value problem in ordinary differential equations

$$y'(t) = f(t, y(t)), \quad y(t_0) = y_0, \tag{1.1}$$

where $y_0 \in R^s$, f: $R \times R^s$ to R^s, is continuous. The approximate solution to (1.1) can be obtained by the multiderivative block method (MDBM) with second order derivatives:

$$y_{n+i} = y_n + h\sum_{j=1}^{k} a_{ij} f_{n+j} + h^2 \sum_{j=1}^{k} b_{ij} f_{n+j}^{(1)} + h\beta_{li} f_n + h^2 \beta_{2i} f_n^{(1)}, \tag{1.2}$$

where $i = 1, \cdots, k$, $y_n \in R^s$, $f_n := f(t_n, y_n) \in R^s$, and $f_n^{(1)} := df(t_n, y_n)/dt \in R^s$ are known vectors. It is proved that there exist $a_{ij}, b_{ij}, \beta_{li}$ and $\beta_{2i}, i, j = 1, 2, \cdots, k$, such that (1.2) converges with order $p = 2k + 2$ (see [1]), and is A-stable for $k \le 5$ (see [5]). To compute the approximate solution $y_{n+j} \doteq y(t_{n+j})$ requires the solution of the following nonlinear equations:

$$y_{n+i} = u_n + h\sum_{j=1}^{k} a_{ij} f_{n+j} + h^2 \sum_{j=1}^{k} b_{ij} f_{n+j}^{(1)}, \quad 1 \le i \le k, \tag{1.3}$$

where $u_n = y_n + h\beta_{li} f_n + h^2 \beta_{2i} f_n^{(1)}$.

Denote y_{n+j} by $y_j, f(t_{n+j}, y_{n+j})$ by $f_j(y_j)$, and $f^{(1)}(t_{n+j}, y_{n+j})$ by $g_j(y_j)$. Then (1.3) becomes

$$y_i = u_n + h\sum_{j=1}^{k} a_{ij} f_j(y_j) + h^2 \sum_{j=1}^{k} b_{ij} g_j(y_j), \quad 1 \le i \le k. \tag{1.4}$$

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There exists a unique solution to (1.4) if and only if the following nonlinear equations have a unique solution (see [2]):

$$y_i = h\sum_{j=1}^{k} a_{ij} f_j(y_j) + h^2 \sum_{j=1}^{k} b_{ij} g_j(y_j), \quad 1 \le i \le k. \tag{1.5}$$

2. Sufficient conditions for the existence and uniqueness of the solution to (1.5)

In this section, we will present sufficient conditions for the existence and uniqueness of the solution to (1.5).

Let $A = (a_{ij})$, $C = (c_{ij}) \in R^{k\times k}$. The Kronnecker product of A and C is defined by

$$A \otimes C = \begin{pmatrix} a_{11}C & a_{12}C & \cdots & a_{1k}C \\ a_{21}C & a_{22}C & \cdots & a_{2k}C \\ \vdots & \vdots & \cdots & \vdots \\ a_{k1}C & a_{k2}C & \cdots & a_{kk}C \end{pmatrix}$$

where

$$a_{ij}C = \begin{pmatrix} a_{ij}c_{11} & a_{ij}c_{12} & \cdots & a_{ij}c_{1k} \\ a_{ij}c_{21} & a_{ij}c_{22} & \cdots & a_{ij}c_{2k} \\ \vdots & \vdots & \cdots & \vdots \\ a_{ij}c_{k1} & a_{ij}c_{k2} & \cdots & a_{ij}c_{kk} \end{pmatrix}.$$

Lemma 2.1. *Let L and D be $m \times m$ matrices, and I_s be the $s \times s$ unit matrix. Then $(LD) \otimes I_s = (L \otimes I_s)(D \otimes I_s)$. Furthermore, if $DL + L^TD$ and D are positive definite, then $\left(DL + L^TD\right) \otimes I_s$ and $D \otimes I_s$ are positive definite.*

Proof. This lemma can be proved directly by the theorems in [3].

Let

$$L = \begin{pmatrix} A & B \\ 0 & I_k \end{pmatrix} \qquad \mathcal{L} = L \otimes I_s = \begin{pmatrix} A \otimes I_s & B \otimes I_s \\ 0 & I_k \otimes I_s \end{pmatrix}$$

where

$$A = (a_{ij})_{k\times k}, \quad B = (b_{ij})_{k\times k}.$$

Denote

$$y = \left(y_1^T, y_2^T, \cdots, y_k^T\right)^T, \quad f(y) = \left(f_1(y_1)^T, f_2(y_2)^T, \cdots, f_k(y_k)^T\right)^T,$$

$$g(y) = g\left(g_1(y_1)^T, g_2(y_2)^T, \cdots, g_k(y_k)^T\right)^T, \quad Y_g = \begin{pmatrix} y \\ h^2 g(y) \end{pmatrix},$$

$$F(Y) = \begin{pmatrix} hf(y) \\ h^2 g(y) \end{pmatrix}$$