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A CLASS OF ASYNCHRONOUS PARALLEL MULTISPLITTING RELAXATION METHODS FOR LARGE SPARSE LINEAR COMPLEMENTARITY PROBLEMS *1)

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Abstract

Asynchronous parallel multisplitting relaxation methods for solving large sparse linear complementarity problems are presented, and their convergence is proved when the system matrices are H-matrices having positive diagonal elements. Moreover, block and multi-parameter variants of the new methods, together with their convergence properties, are investigated in detail. Numerical results show that these new methods can achieve high parallel efficiency for solving the large sparse linear complementarity problems on multiprocessor systems.

Key words: Linear complementarity problem, Matrix multisplitting, Relaxation method, Asynchronous iteration, Convergence theory.

1. Introduction

Consider the linear complementarity problem LCP(M,q):

$$Mz + q \geq 0, \quad z \geq 0, \quad z^T(Mz+q) = 0,$$

where $M = (m_{kj}) \in \mathbb{R}^{n\times n}$ and $q = (q_k) \in \mathbb{R}^n$ are given real matrix and vector, respectively. This problem usually arises in (linear and) convex quadratic programming, in the problem of finding a Nash equilibrium point of a bimatrix (e.g., Cottle and Dantzig [13] and Lemke [25]), and also in a number of free boundary problems of fluid mechanics (e.g., Cryer [17]). Therefore, it has various practical backgrounds. Many efficient iterative methods were established to get a numerical solution of the LCP(M,q) on sequential computer systems, and their convergence properties were discussed in depth (see [1], [10], [13], [14], [16], [17], [23], [24], [25], [26], [28] and [31]). For a systematic and comprehensive study one can refer to Cottle, Pang and Stone [15].

To solve the LCP(M,q) on a high-speed multiprocessor system, we proposed two classes of synchronous multisplitting relaxation methods by successively projecting the unknowns into $\mathbb{R}^n_+ = \{x = (x_1, x_2, \cdots, x_n)^T \mid x_i \geq 0, i = 1, 2, \cdots, \alpha\}$ (see Bai [3]) and by equivalently transforming the LCP(M,q) into a system of fixed-point equations (see Bai and Evans [5] and Bai, Evans and Wang [6]). In a quite different way, Machida, Fukushima and Ibaraki [27] and Bai[4] recently presented and discussed another class of multisplitting iterative methods by implicit splittings of the system matrix. These methods have good parallel computational properties and are suitable for implementing on synchronous parallel computer systems. They can achieve high parallel efficiency provided the task is roughly evenly distributed among all processors.

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However, in practical applications, many problems are natural to be decomposed into subproblems of unequal sizes due to the special physical properties of the original problems. Hence, the above assumption about the balanced distribution of a task does not always hold in actual computations.

To overcome this shortcoming of the abovementioned synchronous parallel multisplitting iterative methods, and to reduce the idle time of each processor, so that the multiprocessor system can achieve high parallel efficiency, in this paper, we further present asynchronous and relaxed variants of the synchronous multisplitting iterative methods proposed in [27] and [4], in accordance with the principle of using sufficiently and communicating flexibly the currently available information. These asynchronous multisplitting relaxation methods can be implemented on MIMD multiprocessor system without any mutual wait among the processors, and hence, they can achieve high parallel computing efficiency in practical applications. When the system matrix $M \in \mathbb{R}^{n\times n}$ is an H_+-matrix , we set up the convergence theories of these new methods under suitable conditions on both the multiple splittings and the relaxation parameters. Moreover, for the convenience of applications, some explicit variants of the new asynchronous multisplitting relaxation methods are presented by making use of the successive overrelaxation technique, and their convergence properties are investigated in detail. With some numerical experiments, we show that these new asynchronous multisplitting relaxation methods can solve large sparse linear complementarity problems on multiprocessor systems with high parallel efficiency.

At last, we remark that this work is also a further development of the asynchronous parallel matrix multisplitting relaxation methods and theories for linear systems of equations in Wang, Bai and Evans [33], Bai, Wang and Evans [9] and Evans, Wang and Bai [20]; In-depth studies on parallel synchronous and asynchronous relaxation methods based on operator projection and fixed-point transformation techniques for solving the large sparse linear complementarity problems can be found in [3], [5], [6], [11], [18] and [29]; Asynchronous variants of the synchronous multisplitting relaxation methods in [3], [5] and [6] were given in [3] and [8], respectively; And generalizations to nonlinear complementarity problem of the synchronous multisplitting relaxation methods in [3] were discussed in [2].

2. Establishments of the New Methods

Without loss of generality, we assume that the considered multiprocessor system consists of α processors, and the host processor may be chosen to be any one of them. For a matrix $M \in \mathbb{R}^{n\times n}$, let $M = B_i + C_i$ $(i = 1, 2, \cdots, \alpha)$ be α Q-splittings and $E_i \in \mathbb{R}^{n\times n}$ $(i = 1, 2, \cdots, \alpha)$ be α nonnegative diagonal matrices satisfying $\sum_{i=1}^{\alpha} E_i = I$ (the $n \times n$ identity matrix). Then the collection of triples $(B_i, C_i, E_i)(i = 1, 2, \cdots, \alpha)$ is called a multisplitting of the matrix M, and the matrices $E_i(i = 1, 2, \cdots, \alpha)$ are called weighting matrices. To describe the new asynchronous multisplitting relaxation methods for the LCP(M,q), we introduce the following necessary notations: $N_0 = \{0, 1, 2, \cdots\}$; for $\forall p \in N_0$, $J(p)$ is a nonempty subset of the number set $\{1, 2, \cdots, \alpha\}$; and for $\forall i \in \{1, 2, \cdots, \alpha\}$ and $\forall p \in N_0$, $s_i(p)$ is an infinite sequence of nonnegative integers. Natural and standard conditions about $J(p)$ and $s_i(p)(i = 1, 2, \cdots, \alpha), p \in N_0$, in the convergence analysis of an asynchronous parallel iteration are:

(1) for $\forall i \in \{1, 2, \cdots, \alpha\}$, the set $\{p \in N_0 | i \in J(p)\}$ is infinite;

$M = (m_{kj}) \in \mathbb{R}^{n\times n}$ is called an H_+-matrix if $m_{kk} > 0(k = 1, 2, \cdots, n)$ and there exist positive reals $w_k(k = 1, 2, \cdots, n)$ such that $\overline{M} = W^{-1}MW$, where $W = \mathrm{diag}(w_1, w_2, \cdots, w_n)$, is a diagonally dominant matrix. In this case, the matrix $M \in \mathbb{R}^{n\times n}$ is also called a generalized diagonal dominant matrix. (See the equivalent definition of H_+-matrix in Section 3.)

$M = B + C$ is called a Q-splitting if B is a Q-matrix. See the definition about a Q-matrix in Section 3.