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COMPUTING EIGENVECTORS OF NORMAL MATRICES WITH SIMPLE INVERSE ITERATION *1)

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Abstract

It is well-known that if we have an approximate eigenvalue $\hat{\lambda}$ of a normal matrix A of order n, a good approximation to the corresponding eigenvector u can be computed by *one* inverse iteration provided the position, say $k_{\max}$, of the largest component of u is known. In this paper we give a detailed theoretical analysis to show relations between the eigenvector u and vector x_k, $k = 1, \cdots, n$, obtained by simple inverse iteration, i.e., the solution to the system $(A - \hat{\lambda} I)x = e_k$ with e_k the kth column of the identity matrix I. We prove that under some weak conditions, the index $k_{\max}$ is of some optimal properties related to the smallest residual and smallest approximation error to u in spectral norm and Frobenius norm. We also prove that the normalized absolute vector $v = |u|/\|u\|_\infty$ of u can be approximated by the normalized vector of $(\|x_1\|_2, \cdots, \|x_n\|_2)^T$. We also give some upper bounds of $|u(k)|$ for those "optimal" indexes such as Fernando's heuristic for $k_{\max}$ without any assumptions. A stable double orthogonal factorization method and a simpler but may less stable approach are proposed for locating the largest component of u.

Key words: Eigenvector, Inverse iteration, Accuracy, Error estimation.

1. Introduction

Let A be a normal matrix of order n. Assume that we have a good approximation $\hat{\lambda}$ to an eigenvalue λ of A, the inverse iteration method

$$(A - \hat{\lambda} I)y_j = z_j, \quad z_{j+1} = y_j/\|y_j\|_\infty$$

is commonly used for computing an eigenvector u of A corresponding λ approximately. In general, the starting vector $z_0 = b$ is chosen at random or to be the vector of all one's and the iteration process converges in several steps [1]. However, there are no practical ways to choose a starting vector b that ensures the rapid convergence, though it is true in theory that one can get an accurate eigenvector to working precision by *a single inverse iteration* if the right vector b is reasonably chosen [7]. In [12], Wilkinson pointed out that for symmetric traditional matrix A, a solution to the homogenous system $(A-\hat{\lambda} I)x = 0$, discarding one of the n equations, say the kth one, will be a good approximation to the eigenvector u provided the kth component $u(k)$ of u is not small. Equivalently, such an approximation, say x_k, can be obtained by one step of inverse iteration $(A - \hat{\lambda} I)x = e_k$ for a properly chosen index k, for example $k = k_{\max}$ corresponding to the largest component $u(k_{\max})$ of u in absolute value. Actually if $u(k)$ is the largest one in absolute value or above average in magnitude, the normalized output $x_k/\|x_k\|$ of a single inverse iteration will yield a residual which archives the *optimal* accuracy in magnitude. (See Corollary 4.1 for details.) It means that the *simple inverse iteration*, a single inverse iteration with right

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vector $b = e_k$, will give an acceptable approximate eigenvector if the index k is chosen well. Therefore there are two related problems that need to be considered: 1) how to locate the largest component $u(k_{\max})$ of the eigenvector u and, 2) if an index k is approximately estimated to $k_{\max}$, how large the component $u(k)$ is or how close it is to $u(k_{\max})$ in absolute value. In [3], an index corresponding to the largest diagonal entry of the inverse of matrix $A - \hat{\lambda}I$ was suggested as an heuristic for choosing the "optimal" index $k_{\max}$. (The index determined by the heuristic will be denoted as k_d in this paper.) Parlett and Dhillon [9] shown that k_d is asymptotically equal to $k_{\max}$ as $\hat{\lambda}$ tends to the eigenvalue λ. In this paper, we will furthermore discuss such problems for a real symmetric or more generally, normal matrix A by a detailed componentwise analysis of the output $x = x_k$. As shown later, under some weak conditions the index $k_{\max}$ is of some optimal properties such that among all normalized vectors x_k^*, $x_{k_{\max}}^*$ achieves the minimum of residuals both in 2-norm and in ∞-norm. In general, for indexes with some optimal properties, for example $k = k_d$, the corresponding component $u(k)$ is the largest one of u with a factor tightly close to one. For those indexes k corresponding to small components $|u(k)|$, the normalized vector x_k^* may be not a good approximation to u, but the position of its largest component in absolute value also implies the position of large component of u, provided $|u(k)|$ is not small enough. On the other hand, the normalized absolute-valued vector $|u|/\|u\|_\infty$ can also be approximated by the normalized vector of the norm vector $(\|x_1\|, \cdots, \|x_n\|)^T$.

Fernando's approach for determining the index k_d is an application of double factorization (a combination of LDU and UDL factorizations) of the nearly singular tridiagonal matrix $A - \hat{\lambda}I$. (Cf. [9] for careful discussions of the relation between the double triangular factorization and the related eigenvector algorithms.) However double factorization is unstable and the slight danger of overflow and/or underflow still exits. We will propose an orthogonal double factorization based upon QR and QL decompositions to determine k_d stably.

This paper is organized as follows: In Section 2, we first review some error bounds of the residual $\|Ax - \tilde{\lambda}x\|_2$ and the error $\|x - u\|_2$ of the approximate eigenvector x computed by a single inverse iteration with respect to the right vector b. As a deduction, error bounds for x_k obtained by simple inverse iteration are also given. In Section 3 we discuss some optimal properties of $x_{k_{\max}}$ that implies information of locating the largest component of u. A lower bound in terms of $u(k_{\max})$ for the component $u(k_d)$ will be given in Section 4, which shows that $u(k_d)$ is always the largest component of u with a factor tightly close to one. We also shown a simpler way to locating largest component of u. The double orthogonal factorization for determining k_d is proposed in Section 5.

Notations. We define by $\{\lambda_j\}$ the set of eigenvalues of matrix A and by $\{u_j\}$ the corresponding eigenvectors with $\|u_j\|_2 = 1$. The eigenvalue λ_i satisfying $|\lambda_i - \hat{\lambda}| = \min_j |\lambda_j - \hat{\lambda}|$ will be simply denoted as λ. Generally, we always assume that λ is uniquely determined, i.e., if $\lambda_j \neq \lambda$, then $|\lambda_j - \hat{\lambda}| > |\lambda - \hat{\lambda}|$. V_λ denotes the eigenspace spanned by the eigenvectors u_i corresponding to $\lambda_i = \lambda$. (The eigenvalue λ may be multiple.) Specially, if λ is a single eigenvalue, $V_\lambda = \text{spann}\{u\}$, where $u = u_i$. It is also assumed that $\hat{\lambda}$ is not an exact eigenvalue of A. x^H means the conjugate transpose of x and, as we have used, $x(k)$ is the k-th component of vector x.

2. A Review on Inverse Iteration

We focus on a single inverse iteration, i.e., an inverse iteration is viewed as a "direct" method for computing approximately eigenvectors rather than an iteration approach. The nature problem is thus that how good the approximation gotten by one iteration

$$(A - \hat{\lambda}I)x = b \tag{2.1}$$

is for a certain right vector b chosen in practical. To that end, let us first review a well-known