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ON FINITE ELEMENT METHODS FOR INHOMOGENEOUS DIELECTRIC WAVEGUIDES *1)

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Dedicated to Professor Zhong-ci Shi on the occasion of his 70th birthday

Abstract

We investigate the problem of computing electromagnetic guided waves in a closed, inhomogeneous, pillared three-dimensional waveguide at a given frequency. The problem is formulated as a generalized eigenvalue problem. By modifying the sesquilinear form associated with the eigenvalue problem, we provide a new convergence analysis for the finite element approximations. Numerical results are reported to illustrate the performance of the method.

Mathematics subject classification: 65N30, 35L15
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1. Introduction

We consider in this paper a closed waveguide defined by a right cylinder with cross section Ω, a bounded, Lipschitz, simply connected polyhedral domain in $\mathbf{R}^2$. The waveguide is filled with inhomogeneous media whose electromagnetic properties are described by the real-valued functions ε and μ. We assume the magnetic permeability $\mu = \mu_0$, the magnetic permeability in vacuum, and the dielectric permittivity ε is piecewise constant and has no variation along the waveguide. More precisely, let $\Omega_1 \subset \Omega$ be an open domain, $\Omega_2 = \Omega \backslash \bar{\Omega}_1$. We assume

$$\varepsilon(x) = \begin{cases} \varepsilon_1\varepsilon_0 & \text{in} \quad \Omega_1, \\ \varepsilon_2\varepsilon_0 & \text{in} \quad \Omega_2, \end{cases}$$

where ε_0 is the dielectric permittivity in vacuum.

The waveguide problem is to find solutions to Maxwell equations which are of the general form

$$\begin{cases} \mathcal{E}(x, x_3, t) = (\mathbf{E}(x), E_3(x))e^{i(\omega t - \beta x_3)} \\ \mathcal{H}(x, x_3, t) = (\mathbf{H}(x), H_3(x))e^{i(\omega t - \beta x_3)} \end{cases} \tag{1.1}$$

where $x \in \Omega$ and the x_3-axis is along the waveguide, $\omega > 0$ is the angular frequency of the guided wave, β is the constant of propagation, $\mathbf{E}$ and $\mathbf{H}$ are electric and magnetic field components in

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the plane of the cross section, and E_3 and H_3 are electric and magnetic components along the waveguide.

With ansatz (1.1), the second order three dimensional Maxwell equations expressed in terms of electric field $(\mathbf{E}, E_3)$ reduce to the following two-dimensional equations (cf. e.g. [11]):

$$\nabla \times (\nabla \times \mathbf{E}) - i\beta \nabla E_3 - (\omega^2 \varepsilon_0 \mu_0)\varepsilon \mathbf{E} = -\beta^2 \mathbf{E} \quad \text{in} \quad \Omega, \tag{1.2}$$

$$\nabla \cdot (\varepsilon \mathbf{E}) - i\beta \varepsilon E_3 = 0 \quad \text{in} \quad \Omega. \tag{1.3}$$

For simplicity, perfect electric conductor boundary conditions are imposed

$$\mathbf{E} \times \mathbf{n} = 0, \quad E_3 = 0 \quad \text{on } \partial\Omega, \tag{1.4}$$

where $\mathbf{n}$ is the unit outer normal to $\partial\Omega$.

Advances in various branches of photonics technologies have established the need for the development of numerical and approximate methods for the analysis of a wide range of waveguide structures that are not amenable to exact analytical studies [5]. The problem (1.2)-(1.3) is an eigenvalue problem. Either ω or β is assumed to be known, and the goal is to find all possible pairs which consist of the other missing constant β or ω and the corresponding field $(\mathbf{E}, E_3)$. The case with a given real-valued β has been extensively studied in the literature (see e.g. [8], [2] and the references therein). More physically relevant case with a given ω to find unknown β is recently studied in [11], in which the eigenvalue problem is studied under the assumption that the frequency ω does not belong to the spectrum of the variational eigenvalue problems associated with the curl-curl and div-grad operators. We remark that since the spectrum of these two operators are generally unknown, this assumption on ω cannot be verified in practical applications.

In this paper we are going to provide a new convergence analysis for the eigenvalue problem (1.2)-(1.3) which removes the restrictions on the frequency ω in [11]. This is achieved by modifying the sesquilinear form associated with the variational formulation of (1.2)-(1.3). The key technical difficulty is the proof of the inf-sup condition of the modified sesquilinear form which allows us to use the general framework for the approximation of the eigenvalue problems developed in [1]. We introduce a finite element method which uses the lowest order Nedelec edge element and standard conforming linear finite element to approximate $(\mathbf{E}, E_3)$, respectively. This choice of finite elements is shown in [11] to exclude spurious modes. Here again the discrete inf-sup condition is proved without any restrictions on the frequency ω and the mesh sizes. We also report several numerical experiments to illustrate the performance of the method studied in this paper.

2. The Continuous Problem

We begin with introducing the Hilbert space $\mathbb{X} = H_0(\text{curl};\Omega) \times H_0^1(\Omega)$ which is equipped with the norm

$$\|(\mathbf{V}, q)\|_{\mathbb{X}} = \|\, \mathbf{V} \,\|_{\text{curl},\Omega} + \|\, q \,\|_{H^1(\Omega)} \qquad \forall (\mathbf{V}, q) \in \mathbb{X}.$$

Here $\|\, \mathbf{V} \,\|_{\text{curl},\Omega} = (\|\, \nabla \times \mathbf{V} \,\|_{L^2(\Omega)}^2 + \|\, \mathbf{V} \,\|_{L^2(\Omega)}^2)^{1/2}$ is the norm of the space $H(\text{curl};\Omega)$ which is defined as the collection of all functions $\mathbf{V}$ in $L^2(\Omega)$ such that $\|\, \mathbf{V} \,\|_{\text{curl},\Omega} < \infty$. $H_0(\text{curl};\Omega)$ consists of functions $\mathbf{V}$ in $H(\text{curl};\Omega)$ whose tangential component $\mathbf{V} \times \mathbf{n}$ vanishes on the boundary $\partial\Omega$.

Set $E_3^{\text{new}} = -i\beta E_3$ in (1.3). To save the notation, E_3 will represent E_3^{new} for the reminder of this paper. Let $k_0^2 = \omega^2 \varepsilon_0 \mu_0$. For $\Lambda > 0$, by adding $\Lambda \mathbf{E}$ on both sides of (1.2), we can reformulate (1.2)-(1.3) with boundary condition (1.4) into the following variational form: