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A Class of Preconditioned TGHSS-Based Iteration Methods for Weakly Nonlinear Systems

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Abstract. In this paper, we first construct a preconditioned two-parameter generalized Hermitian and skew-Hermitian splitting (PTGHSS) iteration method based on the two-parameter generalized Hermitian and skew-Hermitian splitting (TGHSS) iteration method for non-Hermitian positive definite linear systems. Then a class of PTGHSS-based iteration methods are proposed for solving weakly nonlinear systems based on separable property of the linear and nonlinear terms. The conditions for guaranteeing the local convergence are studied and the quasi-optimal iterative parameters are derived. Numerical experiments are implemented to show that the new methods are feasible and effective for large scale systems of weakly nonlinear systems.

AMS subject classifications: 65F10, 65F50

Key words: System of weakly nonlinear equations, GHSS iteration method, local convergence, inner iteration, outer iteration.

1. Introduction

Consider the solution of large sparse system of weakly nonlinear equations of the form

$$Ax = \Phi(x), \quad \text{or equivalently,} \quad F(x) := Ax - \Phi(x) = 0, \tag{1.1}$$

where $A \in \mathbb{C}^{n\times n}$ is a non-Hermitian positive definite matrix, i.e., the Hermitian part $H = (A+A^*)/2$ is positive definite. Here A^* denotes the conjugate transpose of the matrix A. Φ: $\mathbb{D} \subset \mathbb{C}^n \to \mathbb{C}^n$ is a continuously differentiable function defined on the open convex domain $\mathbb{D}$ in the n−dimensional complex linear space $\mathbb{C}^n$. The system of nonlinear equation (1.1) is said to be weakly nonlinear if the linear term Ax is strongly dominant over the nonlinear term $\Phi(x)$ in certain norm. For more details, see [5, 10, 12, 25].

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Large sparse system of nonlinear equations of the form (1.1) arises in many areas of scientific computing and engineering applications, e.g., in finite-difference or sinc discretizations of nonlinear partial differential equations [4,19], in saddle point problems from image processing [9,11] and in collocation approximations of nonlinear integral equation [23], and so on. See [10,18] and referees therein.

For solving weakly nonlinear equations (1.1), the Newton iteration method is an efficient method, which can be described as:

$$x^{(k+1)} = x^{(k)} + s^{(k)}, \quad \text{with} \quad F'(x^{(k)})s^{(k)} = -F(x^{(k)}), \quad k = 0, 1, \cdots,$$

where $F'(x^{(k)})$ is the Jacobian matrix of $F(x^{(k)})$. It's known that the Newton iteration method has quadratic convergence speed if a good initial guess $x^{(0)}$ for the equation $F(x) = 0$ is obtained. To simplify or avoid computation of the Jacobian matrix and reduce the cost of the function evaluation, one can consider many variants in terms of approximate, quasi-update, inner/outer or inexact Newton methods, see [3,5,7,10,26].

Based on the matrix multi-splitting technique, Bai [5,6] present a class of sequential two-stage iteration methods and an efficient parallel generalizations of the sequential two-stage iteration methods. Recently, by making use of separability and dominance between the linear term Ax and the nonlinear term $\Phi(x)$, Bai and Yang [10] present the Picard-HSS and the nonlinear HSS-like iteration methods based on the Hermitian and skew-Hermitian splitting (HSS) for the large sparse non-Hermitian positive-definite system. Furthermore, Bai and Guo [12] established a Newton-HSS iteration method. This method requires one to explicit form of the Jacobian matrix $F'(x^{(k)}) = A - \Phi'(x^{(k)})$ at each iteration step for the current iterate $x^{(k)}$ and solve two linear sub-systems with the coefficient matrices $\alpha I + H(x^{(k)})$ and $\alpha I + S(x^{(k)})$, where α is a prescribed positive parameter, I is the identity matrix. $H(x^{(k)})$ and $S(x^{(k)})$ are the Hermitian and the skew-Hermitian parts of the Jacobian matrix $F'(x^{(k)})$, i.e.,

$$H(x) = H(F'(x)) \quad \text{and} \quad S(x) = S(F'(x)), \quad \forall x \in \mathbb{D}.$$

Here $H(.)$ and $S(.)$ are the Hermitian part and the skew-Hermitian part of the corresponding matrix, respectively. Obviously, the disadvantages of the Newton-HSS iteration methods are often practically prohibitive due to limited computer memory and the admissible computing time. Therefore, authors of [1,17,21,24–26] made further generalizations.

In this paper, based on the generalization of the Hermitian and skew-Hermitian splitting (GHSS) iteration scheme proposed by Benzi [13] and the two-parameter GHSS (TGHSS) [2] iteration method for solving non-Hermitian positive definite linear system, we will first establish a preconditioned TGHSS (PTGHSS) iteration method. Then we propose a class of PTGHSS-based iteration methods, named by Picard-PTGHSS and nonlinear PTGHSS-like iteration methods for solving large sparse weakly nonlinear equations (1.1). Unlike the HSS iteration as the inner iteration, the new iteration methods consist in the fact that the solution of systems with coefficient matrix $\alpha P + S + K$ by inner iterations is made easier, since this matrix is more "diagonally dominant" (loosely speaking) and typically better conditioned than $\alpha P + S$ [13]. Following [2], the quasi-optimal iterative parameters are given. Therefore, the new iteration methods are easy to be implemented.