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General Solutions for a Class of Inverse Quadratic Eigenvalue Problems

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Abstract. Based on various matrix decompositions, we compare different techniques for solving the inverse quadratic eigenvalue problem, where $n \times n$ real symmetric matrices M, C and K are constructed so that the quadratic pencil $Q(\lambda) = \lambda^2 M + \lambda C + K$ yields good approximations for the given k eigenpairs. We discuss the case where M is positive definite for $1 \le k \le n$, and a general solution to this problem for $n+1 \le k \le 2n$. The efficiency of our methods is illustrated by some numerical experiments.

AMS subject classifications: 65F18

Key words: Quadratic eigenvalue problem, inverse quadratic eigenvalue problem, partially prescribed spectral information.

1. Introduction

For $n \times n$ complex matrices M, C and K, the quadratic eigenvalue problem (QEP) involves finding the eigenpairs $(\lambda,\ x)$ such that $Q(\lambda)x = 0$, where

$$Q(\lambda) = Q(\lambda; M, C, K) = \lambda^2 M + \lambda C + K \tag{1.1}$$

is a so-called quadratic pencil defined by M, C and K. The scalars λ and the corresponding nonzero vectors x are the eigenvalues and eigenvectors of the pencil, respectively. It is known that the QEP has $2n$ finite eigenvalues over the complex field, provided that the leading matrix coefficient M is nonsingular. The "direct" problem is of course to find the eigenvalues and eigenvectors when the coefficient matrices M, C and K are given (cf. [5] and references therein), while the inverse quadratic eigenvalue problem (IQEP) is to determine the matrix coefficients M, C and K from a prescribed set of eigenvalues and eigenvectors (cf. [16] and references therein).

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The IQEP has received much attention because of the wide variety of its applications — including structural design [9], control design for second-order systems [6, 16], finite element model updating for damped or gyroscopic systems [7], system identification [1] and inverse problems for damped vibration systems [12]. Some general reviews and extensive bibliographies in this regard can be found in Refs. [3] and [4].

The formulation of an IQEP depends upon the type of eigen-information available, the conditions imposed upon the matrix coefficients, and the techniques used to decompose the matrix constituted by the given eigenvectors. The IQEP studied by Ram & Elhay [17] is for symmetric tridiagonal coefficients where instead of prescribed eigenpairs, two sets of eigenvalues are given. Based on the spectral theory of matrix polynomials, Lancaster *et al.* [8, 11, 13] considered the IQEP with: (1) Hermitian matrices M, C and K, (2) real symmetric matrices M, C and K, and (3) real symmetric positive definite or semi-definite matrices M, C and K, so that the quadratic pencil $Q(\lambda)$ has complete information on the eigenvalues and eigenvectors. We deal with the inverse problem with k given eigenpairs, where M is required to be real symmetric positive definite, and C and K are $n \times n$ real symmetric matrices. For $1 \leq k \leq n$, Yuan *et al.* [18] gave a detailed discussion involving QR decomposition, while for $n+1 \leq k \leq 2n$ Kuo *et al.* [10] studied the general solution to this problem with QR decomposition.

Our main concern is as follows: for a given eigen-information pair (Λ, X), find real symmetric matrices M, C and K where M is positive definite such that

$$MX\Lambda^2 + CX\Lambda + KX = 0 \tag{1.2}$$

is satisfied. Our motivation is to find a more efficient method to solve this problem, and the techniques we investigate below are the Rank Revealing QR (*RRQR*), SVD and UTV factorizations where U and V are orthogonal matrices, while T is an upper-two-diagonal matrix.

Since M, C and K are in $\mathbb{R}^{n\times n}$, we can transform the given complex eigenpairs into real eigenpairs. To facilitate the discussion, let the real eigenpairs constitute the pair of matrices $(\Lambda, X) \in \mathbb{R}^{k\times k} \times \mathbb{R}^{n\times k}$ such that

$$\Lambda = diag\left\{\lambda_1^{[2]}, \cdots, \lambda_l^{[2]}, \lambda_{2l+1}, \cdots, \lambda_k\right\}, \tag{1.3}$$

with

$$\lambda_j^{[2]} = \begin{pmatrix} \alpha_j & \beta_j \\ -\beta_j & \alpha_j \end{pmatrix} \in \mathbb{R}^{2\times 2}, \quad \beta_j \neq 0 \quad \text{for } j = 1, 2, \cdots, l \tag{1.4}$$

and

$$X = \{x_{1R}, x_{1I}, \cdots, x_{lR}, x_{lI}, x_{2l+1}, \cdots, x_k\}, \tag{1.5}$$

where x_{iR} and x_{iI} denote the real and imaginary parts of the corresponding eigenvector, respectively. Then the original eigenpairs can be described by the matrices

$$\tilde{\Lambda} = R^H \Lambda R = diag\{\lambda_1, \lambda_2, \cdots, \lambda_{2l-1}, \lambda_{2l}, \lambda_{2l+1}, \cdots, \lambda_k\}$$