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Superconvergence of Fully Discrete Finite Elements for Parabolic Control Problems with Integral Constraints

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Abstract. A quadratic optimal control problem governed by parabolic equations with integral constraints is considered. A fully discrete finite element scheme is constructed for the optimal control problem, with finite elements for the spatial but the backward Euler method for the time discretisation. Some superconvergence results of the control, the state and the adjoint state are proved. Some numerical examples are performed to confirm theoretical results.

AMS subject classifications: 35B37, 49J20, 65N30

Key words: Superconvergence, finite element method, optimal control problems, parabolic equations, integral constraint.

1. Introduction

The Zienkiewicz-Zhu (ZZ) gradient patch recovery method based on local discrete least-squares fitting [21, 22] is now widely used in engineering practice, due to its robustness in a posteriori error estimates and efficiency in computer implementation. Superconvergence properties of the ZZ patch recovery method have been proven for both linear elements under strongly regular triangular meshes and all popular elements under a rectangular mesh [8, 19].

There has been extensive research on the superconvergence of finite element methods for optimal control problems, mostly focused on the elliptic case. The superconvergence properties of linear and semi-linear elliptic optimal control problems was established in Refs. [15] and [2] respectively, and for finite element approximations of bilinear elliptic optimal control problems [18]. Some superconvergence results for mixed finite element methods applied to elliptic optimal control problems have also been obtained [1, 3, 20].

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In recent years, there has been considerable related research for finite element approximations of parabolic optimal control problems that are frequently met in applications but are much more difficult to handle. A priori and a posteriori error estimates of finite element approximations for parabolic optimal control problems were derived in Refs. [5] and [17], respectively. A priori error estimates for the space-time finite element discretisation of parabolic optimal control problems have been obtained [13, 14], and a posteriori error estimation of spectral methods for parabolic optimal control problems were also investigated [4]. A variational discretisation method for optimal control involving the convection dominated diffusion equation has been considered [6], and superconvergence of a semi-discrete finite element method for parabolic optimal control problems was established [7], although this result has not been implemented in numerical computation. We have previously derived the superconvergence of finite element method for parabolic optimal control problems [16], and to the best of our knowledge there has been little work done on the superconvergence of fully discrete finite element methods for parabolic control problems. The purpose of this article is to investigate the superconvergence of fully discrete finite element approximation for parabolic optimal control problems with integral constraints.

We are interested in the following quadratic parabolic optimal control problem:

$$\begin{cases} \min\limits_{u\in K} \dfrac{1}{2}\displaystyle\int_0^T \left(\|y-y_d\|^2+\|u\|^2\right) dt\,, \\ y_t-\operatorname{div}(A\nabla y)=f+u\,, \qquad x\in\Omega\,,\ t\in J\,, \\ y|_{\partial\Omega}=0\,, \qquad t\in J\,, \\ y(0)=y_0, \qquad x\in\Omega\,, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^2$ with a Lipschitz boundary $\partial\Omega$, and $J=[0,T]$ $(T>0)$. The coefficient $A=(a_{ij}(x))_{2\times 2}\in\left(W^{1,\infty}(\bar{\Omega})\right)^{2\times 2}$ is such that for any $\xi\in\mathbb{R}^2$ we have $(A(x)\xi)\cdot\xi\ge c\mid\xi\mid^2$ with $c>0$.

Let $f\in C(J;L^2(\Omega))$ and $y_0\in H_0^1(\Omega)$, and assume that K is a nonempty closed convex subset in $L^2(J;L^2(\Omega))$, defined by

$$K=\{\, v\in L^2(J;L^2(\Omega)) : \int_0^T\int_\Omega v\,dx\,dt\ge 0\,\}\,.$$

We adopt the standard notation $W^{m,q}(\Omega)$ for Sobolev spaces on Ω with norm $\|\cdot\|_{W^{m,q}(\Omega)}$ and seminorm $|\cdot|_{W^{m,q}(\Omega)}$, set $H_0^1(\Omega)\equiv\left\{v\in H^1(\Omega): v|_{\partial\Omega}=0\right\}$, and denote $W^{m,2}(\Omega)$ by $H^m(\Omega)$. We also denote by $L^s(J;W^{m,q}(\Omega))$ the Banach space of all L^s integrable functions from J into $W^{m,q}(\Omega)$ with norm $\|v\|_{L^s(J;W^{m,q}(\Omega))}=(\int_0^T\|v\|^s_{W^{m,q}(\Omega)}dt)^{1/s}$ for $s\in[1,\infty)$ and the standard modification for $s=\infty$. Similarly, one can define the space $H^l(J;W^{m,q}(\Omega))$ (cf. Ref. [11]). In addition, c or C denotes a generic positive constant.

In Section 2, we define a fully discrete finite element approximation for the model problem, and introduce some intermediate variables and some useful error estimates in Section 3. We derive superconvergence properties in Section 4, and then present some numerical examples in the last section.