East Asian Journal on Applied Mathematics
doi: 10.4208/eajam.180615.080116a

Vol. **6**, No. 1, pp. 89-108
February 2016

Distributed Control of the Stochastic Burgers Equation with Random Input Data

Hyung-Chun Lee* and Yun Nam

Department of Mathematics, Ajou University, Suwon, Korea 16499.

Received 18 June 2015; Accepted (in revised version) 8 January 2016

Abstract. We discuss a control problem involving a stochastic Burgers equation with a random diffusion coefficient. Numerical schemes are developed, involving the finite element method for the spatial discretisation and the sparse grid stochastic collocation method in the random parameter space. We also use these schemes to compute closed-loop suboptimal state feedback control. Several numerical experiments are performed, to demonstrate the efficiency and plausibility of our approximation methods for the stochastic Burgers equation and the related control problem.

AMS subject classifications: 49J20, 76D05, 49B22

Key words: Finite element method, feedback control, stochastic Burgers equation, sparse grid collocation.

1. Introduction

In the study of turbulence phenomena, the Burgers equation provides a simplified and interesting model. For a better understanding of the important problem of the control of turbulence, it has been suggested that involving this equation can be the first step towards application to fluid mechanics problems. Following this strategy, our aim is to study control problems for this equation and to develop computational tools which are powerful enough so that they can be used for the Navier-Stokes equations.

We consider the stochastic Burgers equation with a random coefficient and its distributed control problem, where we want to find an optimal control $f^*(t)$ which minimises the cost functional

$$J(f)=\mathbb{E}\left[\int_0^\infty \Big(||u(t)||^2_{L^2(D)}+\beta||f(t)||^2_{L^2(D)}\Big)dt\right], \tag{1.1}$$

*Corresponding author. *Email addresses:* `hclee@ajou.ac.kr` (H.-C. Lee), `namy0727@ajou.ac.kr` (Y. Nam)

subject to

$$
\begin{aligned}
&\frac{\partial}{\partial t}u(\omega,t,x)-\frac{\partial}{\partial x}\left(a(\omega,x)\frac{\partial}{\partial x}u(\omega,t,x)\right)-u(\omega,t,x)\frac{\partial}{\partial x}u(\omega,t,x)\\
=&f(\omega,t,x)\ \text{in}\ (0,\infty)\times D\,,\\
&\quad u|_{\{0\}\times D}=u_0(x)\,,\qquad u|_{(0,T]\times\partial D}=0\,,
\end{aligned}
\tag{1.2}
$$

where D is $[0,1]$ and $\beta>0$ is a weight. The diffusion coefficient $a(\omega,x)$ and the force term $f(\omega,t,x)$ are random processes on the spatial domain and temporal-spatial domain, respectively. Here $u_0(x)$ is considered to be a deterministic data function and $\mathbb{E}$ denotes an expected value, which is defined as the Lebesque integral in a complete probability space $(\Omega,\mathscr{F},\mathbb{P})$ where Ω is any set, $\mathscr{F}$ is a σ-algebra of subsets of Ω and $\mathbb{P}$ is a probability measure on $\mathscr{F}$ [11,20].

Control problems of the deterministic Burgers equation have been studied by many authors [1,3,4,17,19,21,26,27], and stochastic control problems with additive white noise in Refs. [9,12]. Here we focus on the case of a random process acting on a diffusion coefficient [13,14]. Although the Burgers equation is often considered as the prototype for fluid flow, this equation can also be used as a reasonable mathematical model in other physical contexts such as traffic flow, supersonic flow about airfoils, acoustic transmission, and turbulence in hydrodynamic flows. In brief, the Burgers equation can be regarded as a suitable model for nonlinear wave propagation problems subject to dissipation [15]. Depending on the model problem, this dissipation may result from viscosity, heat conduction, mass diffusion, thermal radiation, chemical reaction, etc.. Thus we also take the viewpoint that the Burgers equation is a variation of the linear heat equation, and an heterogeneous body can be modelled with a *variable thermal conductivity coefficient* (diffusion coefficient) dependent on spatial position. Moreover, when there is a lack of information or uncertainty in the input data, this coefficient can be represented as a random field with estimated statistics.

Our goal here is to develop numerical schemes for a feedback control in minimising a cost function (1.1) subject to the stochastic Burgers equation (1.2). We adopt the finite element method in the spatial discretisation, and sparse grid stochastic collocation in a parameter space where random variables are involved. Later, we illustrate that the sparse grid collocation method is efficient in the optimal choice between the number of nodes and the error in a high-dimensional parameter space to obtain appropriate statistical information. For the optimal control of the Burgers equation, we introduce a feedback law for a linearised equation (i.e. a linear parabolic equation), obtained from a closed-loop system relating the linear quadratic regulator (LQR) theory and the linear quadratic estimation (LQE) problem. The feedback control law from the linearised problem produces the desired extent of stability for the closed-loop nonlinear system, although from the viewpoint of control theory this kind of strategy is actually suboptimal.

In Section 2, we introduce some function spaces, notations and assumptions needed throughout this article. In Section 3, we present a variational formulation for the stochastic burgers equation in order to apply the finite element approximation, and then employ a computable discretisation of the spatial domain in the stochastic sense. We briefly describe