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Nonconforming Finite Element Methods for Wave Propagation in Metamaterials

Changhui Yao [1],*and Lixiu Wang [2]

[1] *School of Mathematics and Statistics, Zhengzhou University, Zhengzhou, 450001, P. R. China.*
[2] *Beijing Computational Science Research Center, Beijing, 100193, P. R. China.*

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Abstract. In this paper, nonconforming mixed finite element method is proposed to simulate the wave propagation in metamaterials. The error estimate of the semi-discrete scheme is given by convergence order $O(h^2)$, which is less than 40 percent of the computational costs comparing with the same effect by using Nédélec-Raviart element. A Crank-Nicolson full discrete scheme is also presented with $O(\tau^2 + h^2)$ by traditional discrete formula without using penalty method. Numerical examples of 2D TE, TM cases and a famous re-focusing phenomena are shown to verify our theories.

AMS subject classifications: 65N30 , 65N15 , 35J25

Key words: Wave Propagation, Metamaterials, Nonconforming Mixed Finite Element, Error Estimates.

1. Introduction

The investigations of wave propagation in Metamaterials have attracted researchers from many areas such as construction of perfect lens, sub-wavelength imaging and cloaking devices. Many numerical simulations have been done on some interesting exotic properties such as negative refractive index and amplification of evanescent waves in Metamaterials which structured electromagnetic composite materials [1, 2].

Generally, numerical simulations in electromagnetic system employ edge finite element method [3–6, 17]. The main advantage is that the spurious solutions can be avoided simultaneously because of the property of curl conforming. In [9], the authors considered three popular dispersives medium models (the isotropic cold plasma medium, one-pole Debye medium and two-pole Lorentz medium) of time-dependent Maxwell's equations in a bounded three-dimensional domain by Nédélec's element, and obtained the optimal order error estimates. In [12], the authors derived the global

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*Corresponding author. *Email addresses:* `chyao@lsec.cc.ac.cn` (C.-H. Yao), `lxwang@csrc.ac.cn` (L.-X. Wang) superconvergence results for semi-discrete scheme. In [13–15], they developed a leapfrog mixed finite element scheme for solving Maxwell's equations. The more merit discrete schemes in time direction can be found in [20, 21]. In [16], the interior penalty discontinuous Galerkin (DG) methods for the time-dependent Maxwell's equations in cold plasma were set up. The above studies are only concentrated on the family of Nédélec's element. However, the Nédélec's element broke down for large-scale computations due to the fact that they could not represent purely TE fields [8]. The others of finite element methods such as C^0-conforming vector nodal finite element methods [7] and nonconforming finite element method [23–25] were also explored by penalty techniques.

The first constructive theoretical and numerical analysis for Maxwell's equations by nonconforming finite element methods can be found in S. C. Brenner's works [23–25], where the Crouzeix-Raviart type triangular nonconforming finite element approximating to two dimensional $curl - curl$ system was studied. And numerical experiments indicated that the traditional weak formula could not lead to a convergence scheme even if the mesh is refined. Therefore, the discrete formula was modified by adding penalty terms, which involved the tangent and normal jumps. The crucial difference is that the piecewise broken $H(curl)\bigcap H(div)$ semi-norm, unlike the piecewise broken H^1 semi-norm for Poisson problem, is too weak to control the jumps. Hence the two terms involving the jumps have to be included in the discrete formula so as to control the consistency error.

Based on the above discussion, it is necessary to reestablish a framework of nonconforming mixed finite element methods approximate the electromagnetic system by traditional discrete scheme without penalty techniques. In [2], the authors summarized a list of ten interesting topics to be explored which concluded the investigations of nonconforming finite element methods. In [19, 22], the authors provided a family of rectangular nonconforming mixed finite element to approximate electromagnetic system, whose theoretical and numerical analysis demonstrated the modeling problems worked successfully. How do these nonconforming mixed finite elements perform in applications?

In this paper, we consider wave propagation in metamaterials by nonconforming mixed finite element method. Re-focusing property of metamaterials can be found clearly. The main advantages conclude the three facts: the first one is the curl conforming in the means of integration for the approximation space of $H(curl,\Omega)$; the second is the transformational relation between the differential operator and the interpolation operator, which will be shown in the following section; the third is the lower computational cost than the corresponding Nédélec's element, which can be reflected by the degrees of freedoms. Another wonderful merit is the superconvergence of consistency term, which leads to overcome the weakness of the discrete norm shown in [23–25]. In the meaning time, we provide the error estimates of the semi-discrete scheme and Crank-Nicolson full discrete scheme for wave propagation model in metamaterials.

The rest of this paper is organized as follows. In Section 2, model of wave propagation presented and a variational formula is provided based on Helmholtz decom-