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Stability of Finite Difference Schemes for Hyperbolic Initial Boundary Value Problems: Numerical Boundary Layers

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Abstract. In this article, we give a unified theory for constructing boundary layer expansions for discretized transport equations with homogeneous Dirichlet boundary conditions. We exhibit a natural assumption on the discretization under which the numerical solution can be written approximately as a two-scale boundary layer expansion. In particular, this expansion yields discrete semigroup estimates that are compatible with the continuous semigroup estimates in the limit where the space and time steps tend to zero. The novelty of our approach is to cover numerical schemes with arbitrarily many time levels.

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1. Introduction and main result

1.1. Introduction

In this article, we are interested in discretizations of transport equations by means of finite difference schemes. When implemented, such numerical schemes require numerical boundary conditions which sometimes can not be deduced from the PDE problem under consideration. This difficulty gives rise to several strategies for which it is crucial to understand whether the resulting numerical schemes is stable and/or consistent. We shall mainly be concerned here with stability issues and refer to [11] for convergence results.

The analysis of numerical boundary conditions for hyperbolic equations is a delicate subject for which several definitions of stability can be adopted. Any such definition relies

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on the choice of a given topology that is a discrete analogue of the norm of some functional space in which the underlying continuous problem is known to be well-posed. The stability theory for numerical boundary conditions developed in [8], though rather natural in view of the results of [15] for partial differential equations, may have suffered from its "technicality". As TREFETHEN and EMBREE [19, chapter 34] say: "[...] *the term GKS-stable is quite complicated. This is a special definition of stability,* [...], *that involves exponential decay factors with respect to time and other algebraic terms that remove it significantly from the more familiar notion of bounded norms of powers*". More precisely, the definition of stability in [8] corresponds to norms of $\ell^2_{t,x}$ type for the numerical solution (t denotes time and x denotes the space variable), while in many problems of evolutionary type one is more used to the $\ell^\infty_t(\ell^2_x)$ topology arising from symmetry and integration by parts arguments. In terms of operator theory, the definition of stability in [8] corresponds to *resolvent estimates* where one eventually proves estimates for the resolvent $(z\,I - T)^{-1}$ of some fixed bounded operator T, while the more familiar notion of bounded norms of powers corresponds to *semigroup estimates* where one wishes to prove that T is power bounded. The links between such resolvent and semigroup estimates have been a rich subject both in the numerical analysis and operator theory communities. We refer for instance to [16, 18] for various results in this direction.

A natural- though delicate- question in the theory of hyperbolic boundary value problems is to pass from GKS type (that is, resolvent) estimates to semigroup estimates. In the context of partial differential equations, this problem has received a somehow final answer in [17], see references therein for historical comments on this problem. In the context of numerical schemes, the derivation of semigroup estimates is not as well understood as for partial differential equations. Semigroup estimates have been derived in [21] for discrete scalar equations, and in [2] for systems of equations. However, the analysis in [21] and [2] only deals with schemes with two time levels, and does not extend as such to schemes with three or more time levels (e.g., the leap-frog scheme). A first attempt to deal with numerical schemes with three or more time levels has been made by one of the authors in [5], but some technical assumptions still exclude applying the theory to, for instance, numerical schemes based on the Adams-Bashforth integration methods.

In this article, we focus on Dirichlet boundary conditions and derive semigroup estimates for a class of numerical schemes with arbitrarily many time levels. The reasons why we choose Dirichlet boundary conditions are twofold. First, these are the only boundary conditions for which, independently of the (stable) numerical scheme that is used for discretizing a scalar transport equation, stability in the sense of GKS is known to hold. The latter result dates back to [10] and is recalled later on. Second, homogeneous Dirichlet boundary conditions typically give rise to numerical boundary layers and therefore to an accurate description of the numerical solution by means of a two-scale expansion. We combine these two favorable aspects of the Dirichlet boundary conditions in our derivation of a semigroup estimate.

The study of numerical boundary layers has received much attention in the past decades, including for nonlinear systems of conservation laws, see for instance [3,6,9]. As far as we know, all previous studies have considered numerical schemes with a three point stencil