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On the Z-Eigenvalue Bounds for a Tensor

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Dedicated to Professor Xiaoqing Jin on the occasion of his 60th birthday

Abstract. In this paper, we first propose a Z_p-eigenvalue of a tensor, which includes the Z_1- and Z_2-eigenvalue as its special case, and then present a Z_p-eigenvalue bound. In particular, we give a Z-spectral radius bound for an irreducible nonnegative tensor via the spectral radius of a nonnegative matrix. The proposed bounds improve some existing ones. Some numerical examples are given to show the validity of the proposed bounds.

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1. Introduction

The Z-eigenvalue problem for a tensor is a useful tool for computing the joint limiting probability distribution of the approximation tensor model of higher-order Markov chains (see e.g., [3, 10]), the PageRank vector in multilinear PageRank models [4, 11], best rank-one approximations in Statistical Data Analysis (e.g., see [6, 7, 20]). We first introduce some definitions and notations, which are the same as in [8].

Let $\mathbb{C}$ ($\mathbb{R}$) be the complex (real) field. An m^{th} order n dimensional tensor in $\mathbb{C}$ is denoted by

$$\mathscr{A} = (a_{i_1\cdots i_m}), \qquad a_{i_1\cdots i_m} \in \mathbb{C}, \qquad 1 \le i_1, \cdots, i_m \le n.$$

A tensor $\mathscr{A}$ is called nonnegative (or, respectively, positive), if $a_{i_1\cdots i_m} \ge 0$ (or, respectively, $a_{i_1\cdots i_m} > 0$) for all $i_1, \cdots, i_m$. A real tensor is called (super-) symmetric [1, 16] if its entries are invariant under any permutation of their indices. We shall denote the set of all m^{th} order n dimensional tensors by $\mathbb{C}^{[m,n]}$, and the set of all nonnegative (or, respectively, positive) m^{th} order n dimensional tensors by $\mathbb{R}_+^{[m,n]}$ (or, respectively, $\mathbb{R}_{++}^{[m,n]}$).

Let $\mathscr{A}$ be an m^{th} order n dimensional tensor and $x = (x_1, \cdots, x_n)^T$ be an n-dimensional vector, we define $\mathscr{A}x^{m-1}$ to be an n-dimensional vector given by

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$$\mathscr{A}x^{m-1} := \left(\sum_{i_2,\cdots,i_m}^{n} a_{i_1\cdots i_m} x_{i_2}\cdots x_{i_m} \right)_{1\le i\le n}. \tag{1.1}$$

Let $\mathbb{P} = \left\{(x_1, x_2, \cdots, x_n)^T | x_i \ge 0\right\}$ be the positive cone, and let the interior of $\mathbb{P}$ be denoted by $int(\mathbb{P}) = \left\{(x_1, x_2, \cdots, x_n)^T | x_i > 0\right\}$. When $y \in \mathbb{P}$ (or $y \in int(\mathbb{P})$), y is said to be a nonnegative (or positive) vector.

The following two definitions of eigenpairs were introduced by Lim [13] and Qi [16], respectively.

Definition 1.1. *Let $\mathscr{A} \in \mathbb{R}^{[m,n]}$. A pair $(\lambda, x) \in \mathbb{C} \times (\mathbb{C}^n\backslash\{0\})$ is called an eigenvalue-eigenvector (or simply eigenpair) of $\mathscr{A}$ if the equation*

$$\mathscr{A}x^{m-1} = \lambda x^{[m-1]} \tag{1.2}$$

holds, where $x^{[m-1]} := (x_1^{m-1}, \cdots, x_n^{m-1})^T$. We call (λ, x) an H-eigenpair if both λ and x are real.

Definition 1.2. *Let $\mathscr{A} \in \mathbb{R}^{[m,n]}$. A pair $(\lambda, x) \in \mathbb{C} \times (\mathbb{C}^n\backslash\{0\})$ is called an E-eigenvalue and an E-eigenvector (or simply an E-eigenpair) of $\mathscr{A}$ if the equations*

$$\mathscr{A}x^{m-1} = \lambda x, \qquad x^T x = 1 \tag{1.3}$$

hold. We call (λ, x) a Z-eigenpair if both λ and x are real. Generally, for $p \ge 1$, a pair $\left(\lambda^{(p)}, x\right) \in \mathbb{R} \times (\mathbb{R}^n\backslash\{0\})$ is called a Z_p-eigenpair of $\mathscr{A}$ if $\left(\lambda^{(p)}, x\right)$ satisfies the equations

$$\mathscr{A}x^{m-1} = \lambda^{(p)} x, \qquad \|x\|_p = 1,$$

where $\|x\|_p = (|x_1|^p + \cdots + |x_n|^p)^{\frac{1}{p}}$.

It is noted that when $p = 1$, a pair $(\lambda^{(1)}, x)$ is called a Z_1-eigenpair (see [2]), which is important for some applications, e.g., for computing the limiting probability distribution in high order Markov chains (e.g. see [8, 10]). For $p = 2$, i.e., Z_2-eigenpair $(\lambda^{(2)}, x)$ is called a Z-eigenpair as in Definition 1.2, which is denoted by (λ, x) for simplicity. By Definition 1.2 of the Z_p-eigenpair it is easy to see that the following lemma holds:

Lemma 1.1. *Let $\mathscr{A} \in \mathbb{R}^{[m,n]}$. If $(\lambda^{(q)}, x)$ is a Z_q-eigenpair, then for any positive number p with $p \ne q$, $\left(\frac{\lambda^{(q)}}{\|x\|_p^{m-2}}, \frac{x}{\|x\|_p}\right)$ is a Z_p-eigenpair.*

In the rest of the paper, without further illustration, we use a Z-eigenvalue to replace a Z_2-eigenvalue.

Definition 1.3. *A tensor $\mathscr{A} = (a_{i_1 i_2\cdots i_m}) \in \mathbb{R}^{[m,n]}$ is called reducible if there exists a nonempty proper index subset $I \subset \{1, \cdots, n\}$ such that*

$$a_{i_1\cdots i_m} = 0, \qquad \forall i_1 \in I, \quad i_2, \cdots, i_m \notin I.$$

If $\mathscr{A}$ is not reducible, then we call $\mathscr{A}$ irreducible.