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A Separable Preconditioner for Time-Space Fractional Caputo-Riesz Diffusion Equations

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Dedicated to Professor Xiaoqing Jin on the occasion of his 60th birthday

Abstract. In this paper, we study linear systems arising from time-space fractional Caputo-Riesz diffusion equations with time-dependent diffusion coefficients. The coefficient matrix is a summation of a block-lower-triangular-Toeplitz matrix (temporal component) and a block-diagonal-with-diagonal-times-Toeplitz-block matrix (spatial component). The main aim of this paper is to propose separable preconditioners for solving these linear systems, where a block ϵ-circulant preconditioner is used for the temporal component, while a block diagonal approximation is used for the spatial variable. The resulting preconditioner can be block-diagonalized in the temporal domain. Furthermore, the fast solvers can be employed to solve smaller linear systems in the spatial domain. Theoretically, we show that if the diffusion coefficient (temporal-dependent or spatial-dependent only) function is smooth enough, the singular values of the preconditioned matrix are bounded independent of discretization parameters. Numerical examples are tested to show the performance of proposed preconditioner.

AMS subject classifications: 65B99, 65M22, 65F08, 65F10

Key words: Block lower triangular, Toeplitz-like matrix, diagonalization, separable, block ϵ-circulant preconditioner, time-space fractional diffusion equations.

1. Introduction

Consider an initial-boundary value problem of the time-space fractional diffusion equation (TSFDE) (see [8])

$$
\begin{aligned}
&{}_0^C D_t^{\alpha} u(x,t) = d(x,t)\frac{\partial^{\beta} u(x,t)}{\partial |x|^{\beta}} + f(x,t), && (x,t)\in(a,b)\times(0,T], && \text{(1.1a)}\\
&u(a,t)=u(b,t)=0, && t\in(0,T], && \text{(1.1b)}\\
&u(x,0)=\psi(x), && x\in[a,b], && \text{(1.1c)}
\end{aligned}
$$

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where $d(x,t)\geq 0$, $u(x,t)$ is unknown to be solved, $f(x,t)$ is source term, $\psi(x)$ is initial condition, ${}_0^C D_t^\alpha u$ is the Caputo's derivative of order $\alpha (0<\alpha<1)$ with respect to t defined by

$$ {}_0^C D_t^\alpha u(x,t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial u(x,s)}{\partial s}(t-s)^{-\alpha} ds. \tag{1.2} $$

$\Gamma(\cdot)$ denotes the gamma function, $\frac{\partial^\beta u(x,t)}{\partial |x|^\beta}$ is the Riesz fractional derivative of order $\beta(1<\beta<2)$ with respect to x defined by

$$ \frac{\partial^\beta u(x,t)}{\partial |x|^\beta} := \sigma_\beta \left({}_a D_x^\beta + {}_x D_b^\beta \right) u(x,t), \quad (x,t)\in(a,b)\times(0,T], \quad \sigma_\beta = -\frac{1}{2\cos(\frac{\pi\beta}{2})} > 0 $$

with the left and the right sided Riemann-Liouville derivatives, ${}_a D_x^\beta u(x,t)$ and ${}_x D_b^\beta u(x,t)$ being defined by

$$ \begin{aligned} {}_a D_x^\beta u(x,t) &= \frac{1}{\Gamma(2-\beta)} \frac{\partial^2}{\partial x^2} \int_a^x \frac{u(\xi,t)}{(x-\xi)^{\beta-1}} d\xi, \\ {}_x D_b^\beta u(x,t) &= \frac{1}{\Gamma(2-\beta)} \frac{\partial^2}{\partial x^2} \int_x^b \frac{u(\xi,t)}{(\xi-x)^{\beta-1}} d\xi, \end{aligned} $$

respectively.

Fractional differential equations are a class of differential equations where the integer-order-derivative terms are replaced by fractional-order derivative. There are several non-equivalent definitions of fractional derivative; see [15, 25, 29]. The Caputo's fractional derivative is often used for time-fractional derivative. The Riemann-Liouville derivatives and the Riesz fractional derivative are often used as space-fractional derivatives. Since closed-form analytical solutions of fractional differential equations are often unavailable especially in the existence of variable coefficients, a lot of useful numerical approximations has been developed for these fractional derivatives; see [2, 6, 9, 13, 19, 21, 23, 28, 35, 36, 40, 42, 44]. Nevertheless, as the fractional differential operators are nonlocal, the discretization of the Caputo's derivative is history-dependent and the discretization of the Riesz's derivative lead to a dense spatial matrix. Therefore, direct solver for the linear systems arising from discretization of TSFDEs requires very high computational complexity when the grid is dense. This motivates us to develop fast solvers for linear systems arising from TSFDEs.

For uniform-grid discretization of the TSFDEs with non-constant coefficients, the resulting coefficient matrix is block lower triangular Toeplitz-like with Toeplitz-like blocks. For such linear systems, the well-known block forward substitution method with Gaussian elimination inner solver requires $\mathcal{O}(MN^2+M^3N)$ operations and $\mathcal{O}(MN+M^2)$ storage, provided that N is the number of blocks in the coefficient matrix and M is the order of each block. The computational cost is quite expensive compared with the number of the unknowns (MN). Recently, Zhao, Jin and Lin [41] proposed to use a combination of time-stepping method and preconditioned generalized minimum residual (PGMRES) method