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Diffusion Limit of 1-D Small Mean Free Path of Radiative Transfer Equations in Bounded Domain

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Abstract. In this paper, we consider the diffusion limit of the small mean free path for the radiative transfer equations, which describe the spatial transport of radiation in material. By using asymptotic expansions, we prove that the nonlinear transfer equation has a diffusion limit as the mean free path tends to zero, and moreover we study the boundary layer problem and mixed layer problem in bounded domain [0,1]. Then we show the validity of their asymptotic expansions relies only on the smoothness of boundary condition, and remove the Fredholm alternative and centering condition.

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Key Words: Transfer equations; asymptotic analysis; diffusion limit; boundary layer; mixed layer.

1 Introduction

The radiative transport in space and the interaction with material medium plays an important role in physical systems (see [1–3]). We ignore the thermal diffusion and material motion, and assume that the material is in local thermodynamic equilibrium. The spatial transport of radiation in a material medium is described by the following equation

$$\partial_t u+\frac{c}{\epsilon}\omega\partial_x u+\frac{\sigma}{\epsilon^2}u=\left(\frac{\sigma}{\epsilon^2}-\sigma_a\right)\overline{u}+\sigma_a v^4, \tag{1.1}$$

and the associated energy balance equation

$$\partial_t v+\frac{\epsilon\sigma_a}{C_h}(\beta v^4-\overline{u})=0. \tag{1.2}$$

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Here $u(t,x,\omega)$ defines the specific intensity at time $t\geq 0$, location $x\in[0,1]$. Among them $\omega=cos\theta$, θ is the angle between the direction of travel of the photon and the positive x-axis. The scalar density is defined as

$$\overline{u}(t,x)=\frac{1}{2}\int_{-1}^{1}u(t,x,\omega)\mathrm{d}\omega, \tag{1.3}$$

where c is the vacuum speed of light. $\sigma=\sigma_a+\sigma_s$ is the transport coefficient, $\sigma_s>0$ is the scattering coefficient, $\sigma_a>0$ is the absorption coefficient, $\epsilon/\sigma>0$ is the mean free path, $\beta>0$ is called the Stefan-Boltzmann constant, v is material temperature, $C_h>0$ is the pseudo-heat capacity.

The computational methods, asymptotic analysis and approximate models of radiative transport equations (1.1)-(1.2) are the focus of scholars such as Pomraning [3], Larsen [2,4,5], Bowers and Wilson [1], Anistratov and Larsen [6], Adams and Larsen [7], Morel [8], Roberts and Anistratov [9]. However, since the mathematically results are difficult to solve, relevant results are few. To our knowledge, there are only two works as follows:

Guo and Han [10] study the well-posedness and initial layer of equations (1.1)-(1.2) in $\mathbb{R}^3$ with initial conditions

$$u(0,x,\omega)=u^0(x,\omega),\quad v(0,x)=v^0(x),\qquad x\in\mathbb{R}^3,\ \omega\in\mathbb{S}^2\subset\mathbb{R}^3,$$

and then they also [11] study the well-posedness and initial-boundary layer of equations (1.1)-(1.2) in $[0,1]$ with absorbing boundary conditions

$$\begin{aligned}&u(0,x,\omega)=u^0(x,\omega),\ v(0,x)=v^0(x),\qquad x\in[0,1],\ \omega\in[-1,1],\\&u(t,x,\omega)|_{x=0}=u_L(t,\omega),\qquad \omega>0,\\&u(t,x,\omega)|_{x=1}=u_R(t,\omega),\qquad \omega<0.\end{aligned}$$

Motivated by their results, we consider two different initial boundary conditions, which cause different kinds of layers. Firstly, we introduce the following initial boundary condition:

$$u(0,x,\omega)=u^0(x),\ v(0,x)=v^0(x),\qquad x\in[0,1], \tag{1.4}$$

$$u_L(t,x,\omega)=u(t,x,\omega)\,|_{x=0}=\begin{cases}A_L(0), & t=0,\\ A_L(t,\omega), & t>0,\ \omega>0,\end{cases} \tag{1.5}$$

$$u_R(t,x,\omega)=u(t,x,\omega)\,|_{x=1}=\begin{cases}A_R(0), & t=0,\\ A_R(t,\omega), & t>0,\ \omega<0.\end{cases} \tag{1.6}$$

Secondly, we focus on the initial boundary condition as follows

$$u(0,x,\omega)=u^0(x,\omega),\ v(0,x)=v^0(x),\qquad x\in[0,1], \tag{1.7}$$

$$u_L(t,x,\omega)=u(t,x,\omega)\,|_{x=0}=A_L(t,\omega),\qquad t>0,\ \omega>0, \tag{1.8}$$