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On Characterization of Nonuniform Tight Wavelet Frames on Local Fields

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Abstract. In this article, we introduce a notion of nonuniform wavelet frames on local fields of positive characteristic. Furthermore, we gave a complete characterization of tight nonuniform wavelet frames on local fields of positive characteristic via Fourier transform. Our results also hold for the Cantor dyadic group and the Vilenkin groups as they are local fields of positive characteristic.

Key Words: Nonuniform wavelet frame, tight wavelet frame, Fourier transform. local field.

AMS Subject Classifications: 43A70, 11S85, 42C40, 42C15

1 Introduction

Frames in a Hilbert space was originally introduced by Duffin and Schaeffer [6] in the context of non-harmonic Fourier series. In signal processing, this concept has become very useful in analyzing the completeness and stability of linear discrete signal representations. Frames did not seem to generate much interest until the ground-breaking work of Daubechies et al. [3]. They combined the theory of continuous wavelet transforms with the theory of frames to introduce wavelet (wavelet) frames for $L^2(\mathbb{R})$. Since then the theory of frames began to be more widely investigated, and now it is found to be useful in signal processing, image processing, harmonic analysis, sampling theory, data transmission with erasures, quantum computing and medicine. Today more applications of the theory of frames are found in diverse areas including optics, filter banks, signal detection and in the study of Bosev spaces and Banach spaces. We refer [4,5] for an introduction to frame theory and its applications.

Tight wavelet frames are distinct from the orthonormal wavelets because of redundancy. By relinquishing orthonormality and permitting redundancy, the tight wavelet frames turn out to be significantly easier to construct than the orthonormal wavelets. In

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applications, tight wavelet frames provide representations of signals and images where repetition of the representation is favored and the ideal reconstruction property of the associated filter bank algorithm, as in the case of orthonormal wavelets is kept.

A field K equipped with a topology is called a local field if both the additive and multiplicative groups of K are locally compact Abelian groups. For example, any field endowed with the discrete topology is a local field. For this reason we consider only non-discrete fields. The local fields are essentially of two types (excluding the connected local fields $\mathbb{R}$ and $\mathbb{C}$). The local fields of characteristic zero include the p-adic field $\mathbb{Q}_p$. Examples of local fields of positive characteristic are the Cantor dyadic group and the Vilenkin p-groups. Even though the structures and metrics of local fields of zero and positive characteristics are similar, their wavelet and multiresolution analysis theory are quite different. For more details we refer to [1].

The local field K is a natural model for the structure of wavelet frame systems, as well as a domain upon which one can construct wavelet basis functions. There is a substantial body of work that has been concerned with the construction of wavelets on K, or more generally, on local fields of positive characteristic. For example, Jiang et al. [9] pointed out a method for constructing orthogonal wavelets on local field K with a constant generating sequence and derived necessary and sufficient conditions for a solution of the refinement equation to generate a multiresolution analysis of $L^2(K)$. Shah and Debnath [10] have constructed tight wavelet frames on local fields of positive characteristic using the extension principles. As far as the construction of wavelet frames on K via Fourier transforms is concerned, Li and Jiang [8] have established a necessary condition and a set of sufficient conditions for the system

$$\Big\{\psi_{j,k} =: q^{j/2}\psi\big(\mathfrak{p}^{-j}x-u(k)\big) : j,k\in\mathbb{N}_0\Big\} \tag{1.1}$$

to be a frame for $L^2(K)$. These studies were continued by Shah and his colleagues in series of papers [11–15].

Motivated and inspired by the above work, we provide the complete characterization of nonuniform tight wavelet frames on local fields of positive characteristic by means of Fourier transform technique. The paper is tailored as follows. In section 2, we discuss some basic facts about local fields of positive characteristic including the notion of nonuniform wavelet frames on local fields of positive characteristic. In section 3, we provide the complete characterization of nonuniform tight wavelet frames on local fields of positive characteristic by using the machinery of Fourier transform.

2 Basic Fourier analysis on local fields

Let K be a field and a topological space. Then K is called a local field if both K^+ and K^* are locally compact Abelian groups, where K^+ and K^* denote the additive and multiplicative groups of K, respectively. If K is any field and is endowed with the discrete topology, then K is a local field. Further, if K is connected, then K is either $\mathbb{R}$ or $\mathbb{C}$. If K is not connected,