J. Math. Study
doi: 10.4208/jms.v51n3.18.03

Vol. **51**, No. 3, pp. 294-308
September 2018

Gevrey Regularity of the Global Attractor for Damped Forced KdV Equation on the Real Line

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Received August 17, 2017; Accepted March 17, 2018

Abstract. We consider here a weakly damped KdV equation on the real line with forcing term that belongs to some Gevrey space. We prove that the global attractor is also contained into such a space of analytic functions.

AMS subject classifications: 35Q53, 37L50, 35B41

Key words: Dissipative KdV equation, global attractor, Gevrey regularity.

1 Introduction

1.1 Global attractor for damped forced KdV equation on the real line

This article is concerned with regularity issues for the global attractor of a weakly damped, forced Korteweg-de Vries (KdV) equation that reads

$$u_t+\gamma u+u_{xxx}+uu_x=f; \tag{1.1}$$

here $\gamma>0$ is the positive damping parameter and the forcing term f does not depends on t. Throughout this article we will assume that f is a real analytic function that belongs to some Gevrey space G^{τ_0} (see the precise definition of Gevrey spaces below). The unknown u is a function from $\mathbb{R}_t\times\mathbb{R}_x$ into $\mathbb{R}$. This article partakes of the infinite-dimensional dynamical system theory (see [10,14,18,24]). For weakly damped dispersive equations as (1.1) a first issue concerns the existence of a global attractor. Indeed, supplemented with an initial data in $H^\rho(\mathbb{R})$, the solution map $S_\rho(t)u_0=u(t)$ defines a semigroup in $H^\rho(\mathbb{R})$. When it exists, the global attractor $\mathcal{A}_\rho$ is a compact invariant attracting set in $H^\rho(\mathbb{R})$ (see [24]). For KdV equations on the whole line, P. Laurençot [13] proved the existence of a global attractor $\mathcal{A}_2$ for the semigroup in $H^2(\mathbb{R})$, while R. Rosa [19] proved the existence of a global attractor $\mathcal{A}_1$ for the semigroup in $H^1(\mathbb{R})$ by the so-called *energy*

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method (see [16] and the references therein). Eventually, in [8] the authors proved that the semigroup $S_0(t)$ in $L^2(\mathbb{R})$ has a compact global attractor $\mathcal{A}_0$ that is smooth, i.e. that if $f\in L^2(\mathbb{R})$ then $\mathcal{A}_0\subset H^3(\mathbb{R})$. As a consequence of this result we have that $\mathcal{A}_0=\mathcal{A}_1=\mathcal{A}_2$. We set $\mathcal{A}=\mathcal{A}_0$ for this global attractor in the sequel. Besides we know that if f is assumed smooth then $\mathcal{A}$ is a subset of $\cap_m H^m(\mathbb{R})$. We also refer to [25] for the study weakly damped forced KdV equations below $L^2(\mathbb{R})$ by the I-method of J. Colliander, M. Keel, G. Staffilani, H. Takaoka and T. Tao (see [22] and the references therein).

We address in this article the following issue: is the global attractor $\mathcal{A}$ included in a set of analytic functions if the forcing term belongs to some Gevrey space? For weakly damped dispersive equations the answer is positive for periodic $1D$ nonlinear Schrödinger equations. This result is due to M. Oliver and E. S. Titi [17] and has some important consequences. The first one is that the Faedo-Galerkin approximation of a solution included in the global attractor converges exponentially fast. The second one is in estimating the number of determining nodes for solutions in the global attractor. This issue for KdV equation was still open. Our main result is as follows

Theorem 1.1. *Assume that the forcing term f belongs to some Gevrey space G^{τ_0}. Then there exists a τ that depends on γ,f such that the global attractor $\mathcal{A}$ is a bounded subset of G^τ.*

For the periodic KdV equations, the regularity of the attractor was proved in [15], [6]. For the Gevrey regularity issue in this periodic setting we refer to [7]. Besides, the study of the initial value problem for KdV equation, from the pioneering works [2,20,23] to more recent methods by C. Kenig, G. Ponce and L. Vega [11], has been boosted by the work of J. Bourgain [3] (see [5,12,22] and the references therein). Moreover the so-called Bourgain method was used in the last decade in [1,9,21] to tackle the initial value problem for KdV equations in Gevrey spaces and other issues related to the analyticity in space of solutions in the conservative case $\gamma=0$ and $f=0$. Therefore our strategy here is to combine the splitting method used in [8] with the use of Bourgain-Gevrey spaces. We consider a complete trajectory that is included in the global attractor, i.e. that is defined for all (positive and negative) times and bounded. We split the solution into a low frequency and a high frequency part, choosing the cut-off N large enough; we then introduce an auxiliary problem whose solution approximates the high frequency part of the trajectory in $L^2(\mathbb{R})$ for large times, and whose initial data belongs to some Gevrey space; we then prove the persistence of the Gevrey regularity for the solution of the auxiliary problem. We conclude shifting backward in time the solution of the auxiliary problem.

The outline of the article is the following. In the remaining of Section 1 we introduce the required mathematical framework for Bourgain-Gevrey spaces and then prove some useful bilinear estimates in these spaces. Section 2 is devoted to the proof of the Theorem 1.1 as described in the previous paragraph. In a last Section we discuss the size of the global attractor in Gevrey spaces G^τ for small τ.