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A New Triangular Spectral Element Method II: Mixed Formulation and *hp*-Error Estimates

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Abstract. Mixed triangular spectral element method using nodal basis on unstructured meshes is investigated in this paper. The method is based on equivalent first order system of the elliptic problem and rectangle-triangle transforms. It fully enjoys the tensorial structure and flexibility in handling complex domains by using nodal basis and unstructured triangular mesh. Different from the usual Galerkin formulation, the mixed form is particularly advantageous in this context, since it can avoid the singularity induced by the rectangle-triangle transform in the calculation of the matrices, and does not require the evaluation of the stiffness matrix. An *hp* a priori error estimate is presented for the proposed method. The implementation details and some numerical examples are provided to validate the accuracy and flexibility of the method.

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1. Introduction

The spectral element method (SEM) (or *hp* finite element method) [22] integrates the unparalleled accuracy of a spectral method and the geometric flexibility of a finite element method, and also enjoys a high-level parallel computer architecture. As such, it plays an exceedingly important part in large-scale simulations [4,8,13,14]. For a long time, we saw SEM through building blocks of quadrilaterals and hexahedra with tensorial structures (QSEM) [4, 8, 22]. The use of tensorial nodal basis functions in a QSEM substantially facilitates both the implementation (e.g., the imposition of continuity across elements) and analysis, as many numerical tools and analysis arguments in one dimension can be

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directly transplanted to multiple dimensions. However, QSEM usually requires the same degree of freedom (DoF) on each element, so it may lose the p-adaptive capability.

In the past two decades, much progress has been made in developing triangular or tetrahedral SEM (TSEM) on unstructured meshes. There are two noticeable trends in designing TSEM. The first is built upon approximation by orthogonal basis related to the collapsed Duffy's transform [9, 10, 13, 16, 24, 26] and its important variant [17, 18, 23]. The second is based on approximation by nodal basis on special nodal points [6, 11, 12, 21, 28]. Here, we elaborate more on the former approach. Firstly, the spectral approximation in triangle using polynomials was much studied (cf. [3, 9, 11, 19, 20, 27, 28]). Recently, some research efforts have been paid to the non-polynomial spectral approximations in triangle/tetrahedron [5, 16, 18, 24]. By using some rectangle-triangle transforms, these spectral methods generate rational or irrational basis functions in triangle from standard tensorial basis functions in rectangle. Two typical rectangle-triangle transforms: Duffy's transform and one-to-one transform (cf. [18]) are frequently adopted. One argument against the Duffy's transform is that the mapped interpolation points are unfavourably clustered near the singular vertex of the triangle. The situation is even severer in the three-dimensional case. To obtain a better distribution of the mapped interpolation points, a new one-to-one transform is designed by pulling one side of the triangle to two sides of the rectangle. As long as the development of the spectral approximations in triangles, more and more attention has been paid on corresponding TSEM. Although the new transform has weaker singularity than the Duffy's transform, it also leads to singular integrand in the calculation of stiffness matrix (cf. [18, 23]). Either a mode basis (cf. [18, 23]) or modified nodal basis [17] is used to handle the singularity. Nevertheless, the special basis functions increase the difficulty in extending to multi-domain cases. Usually, some other techniques (e.g. motar finite element [2, 15]) need to be employed.

This paper is the second of a series on developing TSEM based on the transform [18]. In the first paper [23], a detailed analysis of the logarithmic singularity induced by the transform was conducted and an accurate and stable method to handle such singularities by using mode basis was implemented. Here, we continue to develop a flexible nodal TSEM more applicable to multi-domain cases. The new TSEM is drawn on a mixed formulation using non-polynomial spectral approximations on triangles. Both Duffy's transform and the one-to-one transform can be used to generate non-polynomial basis functions for the method. The main feature of this method is that it is unnecessary to deal with the consistency condition and no singularity will appear in the calculation of the discrete matrices. Actually, the mixed formulation does not involve the stiffness matrix and the singularity in the calculation of other matrices can be eliminated by the Jacobian. Although the mixed formulation introduces a new auxiliary variable, it can be efficiently removed from the discrete linear system due to the fact that the approximated mass matrix is naturally diagonal even in the variable coefficient case. Another main problem to form a spectral element method using non-polynomial spectral approximations in triangles is how to construct and implement a continuous approximation space. We introduce different strategies according to the adopted transform. The proposed TSEM fully enjoys the tensorial structure as QSEM. Hence an efficient implementation can be expected. In theoretical aspect, the H^1-