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A DIAGONALLY-IMPLICIT TIME INTEGRATION SCHEME FOR SPACE-TIME MOVING FINITE ELEMENTS*

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Abstract

In this paper, we analyze and provide numerical experiments for a moving finite element method applied to convection-dominated, time-dependent partial differential equations. We follow a method of lines approach and utilize an underlying tensor-product finite element space that permits the mesh to evolve continuously in time and undergo discontinuous reconfigurations at discrete time steps. We employ the TR-BDF2 method as the time integrator for piecewise quadratic tensor-product spaces, and provide an almost symmetric error estimate for the procedure. Our numerical results validate the efficacy of these moving finite elements.

Mathematics subject classification: 65M55, 65F10
Key words: TR-BDF2, Moving finite elements, Method of characteristics, Convection-dominated, Moving mesh methods, Error analysis.

1. Introduction

The method of lines is an efficient approach for computing numerical solutions to parabolic partial differential equations by converting these problems into systems of ordinary differential equations. This provides a great deal of flexibility in how the solution may be computed, as the time discretization then becomes independent of the spatial discretization. For finite element methods, the spatial dimensions are discretized in the usual way, leading to a semi-discrete problem that is subsequently propagated in time by numerical integration.

When dealing with convection-dominated problems, the spatial discretization can be chosen to evolve continuously in time, which allows the finite element mesh to continuously track moving structures in the solution such as steep sweeping fronts [1–3]. These moving finite elements can lead to remarkably improved stability in computing a solution, with respect to the length of permissible time steps [4, 5]. While the literature of moving mesh finite element methods is expanding rapidly, rigorous error analysis of these methods is still relatively unknown.

In [6,7], tensor-product finite element spaces compatible with a method of lines discretization were introduced that allowed these moving finite element solutions to be studied in a space-time finite element framework. As a result, these papers established symmetric error estimates for these finite element solutions of arbitrary order when the numerical time integrator belongs to

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a particular class of fully implicit collocation methods. The first symmetric error estimate is proven in [8] for semi-discrete moving finite elements,

$$\|u - u_h\| \leq C \inf_{\chi \in \mathcal{V}_h} \|u - \chi\|, \tag{1.1}$$

using a mesh-dependent energy semi-norm, $\|\cdot\|$, where u is the true solution, u_h is the finite element solution, and $\mathcal{V}_h$ is a (tensor-product) moving finite element space [6,7]. Other error analyses include [9] that bound the error in alternative energy norms, [10] for mixed finite element methods, and [11] for moving finite difference schemes.

In this paper, we consider the effects of employing a time integrator that does not belong to previous classes of collocation methods. This is a valuable modification because the collocation methods in past works are fully implicit and couple all intermediate stages of each time step, significantly increasing the computational complexity when using higher order quadrature. We consider the second-order and diagonally-implicit time integrator, TR-BDF2, introduced in [12,13], and using piecewise quadratic tensor-product finite element spaces to discretize the problem. This time integration scheme is known for its favorable stability properties [14,15], which motivates its study in the present context as moving finite element discretizations often lead to stiff systems of ODEs [4,5]. In Section 3, we prove an error estimate like (1.1) with an additional term corresponding to the truncation error of TR-BDF2.

This work largely builds on the analyses in [6,16], where parts of the preliminary analysis are given in more detail. This paper is organized as follows: in Section 2, we describe the model equation, the piecewise quadratic tensor-product finite element space, and some preliminary results. In Section 3, a space-time moving finite element method using TR-BDF2 time integration is proposed and an error estimate for the finite element solution is proven. We note that the proposed scheme is a simple discretization resulting directly from a finite element discretization in space and the method of lines to discretize the time variable; many moving mesh methods fit into the framework of our analysis without modification, as discussed in section 3. Section 4 describes and reports some numerical experiments that validate the efficacy of these moving finite element methods.

2. Preliminary Results

The model problem used in this error analysis is the linear convection-diffusion-reaction equation. The spatial domain, Ω, is assumed to be a simply connected set in $\mathbb{R}^d$, where $d = 1, 2$, or 3, with boundary $\partial\Omega$. The time domain is a finite interval, $(0, T]$, and the space-time domain is given by $\mathcal{F} \equiv \Omega \times (0, T]$.

Let a, b, c, and f be smooth and bounded functions defined on $\mathcal{F}$ such that there exist constants $\bar{a} > 0$ and $\bar{c} \geq 0$ with $a \geq \bar{a}$ and $c \geq \bar{c}$ on Ω, and let g be integrable on $\partial\Omega$. Let u_0 be a given initial condition for the solution on $\mathcal{F}$ and let n denote the outward unit normal vector to the boundary $\partial\Omega$. The solution to the differential equation, denoted by u, is the function that satisfies

$$u_t - \nabla \cdot (a\nabla u) + b \cdot \nabla u + cu = f, \qquad \text{in } \mathcal{F}, \tag{2.1}$$

$$a\nabla u \cdot n = g, \qquad \text{on } \partial\Omega \times (0, T], \tag{2.2}$$

$$u(x, 0) = u_0(x), \qquad \text{for } x \text{ in } \Omega.$$