Analysis in Theory and Applications
Anal. Theory Appl., Vol. **34**, No. 4 (2018), pp. 297-305
DOI: 10.4208/ata.OA-2018-0003

Approximation by a Complex Post-Widder Type Operator

Sorin G. Gal[1,*] and Vijay Gupta[2]

[1] *Department of Mathematics and Computer Science, University of Oradea, Str. Universitatii No. 1, 410087 Oradea, Romania*
[2] *Department of Mathematics, Netaji Subhas Institute of Technology, Sector 3 Dwarka, New Delhi-110078, India*

Received 24 January 2018; Accepted (in revised version) 2 July 2018

Abstract. In the present article, we deal with the so-called overconvergence phenomenon in $\mathbb{C}$ of a slightly modified Post-Widder operator of real variable, that is with the extension of its approximation properties from the real axis in the complex plane. In this sense, error estimates in approximation and a quantitative Voronovskaya-type asymptotic formula are established.

Key Words: Real and complex Post-Widder type operator, overconvergence phenomenon, approximation estimate, Voronovskaya-type result, exact error estimation.

AMS Subject Classifications: 41A25, 41A30, 30E10

1 Introduction

In the case of real functions, in e.g., [2], Chapter 9, the slightly modified Post-Widder operator given by

$$P_n(f;x)=\frac{1}{(n-1)!}\cdot\left(\frac{n}{x}\right)^n\int_0^{+\infty}t^{n-1}e^{-nt/x}f(t)dt$$

is considered, where $f:[0,+\infty)\to\mathbb{R}$, $x>0$.

It is clear that

$$P_{n+1}(f;x)=\frac{1}{n!}\cdot\left(\frac{n+1}{x}\right)^{n+1}\int_0^{+\infty}t^ne^{-(n+1)t/x}f(t)dt$$

*Corresponding author. *Email addresses:* `galso@uoradea.ro` (S. G. Gal), `vijaygupta2001@hotmail.com` (V. Gupta)

and passing here from the discrete parameter n to a continuous parameter $s \geq 1$, we can consider the form (after the change of variable $w = t/x$)

$$\begin{aligned}\overline{P}_s(f;x) &= \frac{1}{\Gamma(s+1)} \cdot \left(\frac{s+1}{x}\right)^{s+1} \int_0^{+\infty} t^s e^{-(s+1)t/x} f(t)dt \\ &= \frac{(s+1)^{s+1}}{\Gamma(s+1)} \cdot \int_0^{+\infty} e^{-(s+1)w} w^s f(wx)dw.\end{aligned}$$

Denoting $e_i(x) = x^i$, $i = 0,1,2$, according to [2], Chapter 9 (see, also [8]) we have

$$\begin{aligned}&\overline{P}_s(e_0;x) = 1, \quad \overline{P}_s(e_1;x) = x, \quad \overline{P}_s(e_2;x) = x^2 + \frac{x^2}{s+1}, \\ &\overline{P}_s((e_1 - x)^2;x) = \frac{x^2}{s+1},\end{aligned}$$

and

$$\overline{P}_s(e_k;x) = \frac{(s+1)\cdot\cdots\cdot(s+k)}{(s+1)^k} \cdot x^k \quad \text{for all} \quad s \geq 1, \quad k \in \mathbb{N}. \tag{1.1}$$

Remark 1.1. In the paper [1] (see also [9], pp. 287), the original Post-Widder operator given by the formula

$$\begin{aligned}L_n(f)(x) &= \frac{1}{n!} \left(\frac{n}{x}\right)^{n+1} \int_0^\infty e^{-nu/x} u^n f(u)du \\ &= \frac{n^{n+1}}{n!} \int_0^\infty e^{-nv} v^n f(vx)dv \quad x > 0, \quad f \in C[0,+\infty),\end{aligned}$$

is studied. Note that simple calculations lead us to

$$\begin{aligned}&L_n(e_0;x) = 1, && L_n(e_1;x) = x + \frac{x}{n}, \\ &L_n(e_2;x) = \left(1 + \frac{1}{n}\right)\left(1 + \frac{2}{n}\right)x^2, && L_n((e_1 - x)^2;x) = x^2\left(\frac{1}{n} + \frac{2}{n^2}\right).\end{aligned}$$

In any case, the original Post-Widder operators $L_n(f;x)$, do not reproduce the linear functions as the modified ones $\overline{P}_s(f;x)$ do.

The overconvergence phenomenon, that is the extension of approximation properties of the positive and linear operators from the real axis in the complex plane, is an intensively studied topic in approximation theory. Thus, for example, the first author estimated the approximation properties of many complex operators in the book [3], while some other complex operators of Durrmeyer type have been discussed in, e.g., [4, 6, 7] and [5], to mention only a few.

In the present paper, we study the approximation properties of a complex operator $\overline{P}_s$ of Post-Widder type.