Commun. Comput. Phys.
doi: 10.4208/cicp.OA-2018-0051

Vol. **25**, No. 4, pp. 988-1009
April 2019

A Simplified Artificial Compressibility Flux for the Discontinuous Galerkin Solution of the Incompressible Navier-Stokes Equations

Fan Zhang[1,2], Hansong Tang[2], Jian Cheng[3] and Tiegang Liu[1,*]

[1] *School of Mathematics and Systems Science, Beihang University, Beijing, 100091, P.R. China.*
[2] *Department of Civil Engineering, City College, City University of New York, NY 10031, USA.*
[3] *Institute of Applied Physics and Computational Mathematics, Beijing, 100088, P.R. China.*

Communicated by Kun Xu

Received 23 February 2018; Accepted (in revised version) 15 May 2018

Abstract. The discontinuous Galerkin (DG) method has attained increasing popularity for solving the incompressible Navier-Stokes (INS) equations in recent years. In this work, we present a novel DG discretization for solving the two-dimensional INS equations in which the inviscid term of the INS equations is split into two parts, the Stokes operator and the nonlinear convective term, and treated separately. The Stokes operator is discretized using the artificial compressibility flux which is provided by the (exact) solution of a Riemann problem associated with a local artificial compressibility perturbation of the Stokes system, while the nonlinear term is discretized in divergency form by using the local Lax-Friedrichs fluxes; thus, local conservativity is inherent. Unlike the existing artificial compressibility flux for the DG discretization of the INS equations which needs to solve a Riemann problem for a nonlinear system by numerical iteration, the separate treatment of the nonlinear term from the Stokes operator makes the Riemann problem become linear and can be solved explicitly and straightforwardly, therefore no iterative procedure is further required. A number of test cases with a wide range of Reynolds number are presented to assess the performance of the proposed method, which demonstrates its potential to be an alternative approach for high order numerical simulations of incompressible flows.

AMS subject classifications: 65M60, 65M99, 35L65

Key words: Discontinuous Galerkin methods, Navier-Stokes equations, incompressible flows, artificial compressibility flux.

*Corresponding author. *Email addresses:* `liutg@buaa.edu.cn` (T. G. Liu), `zhangfan1990@buaa.edu.cn` (F. Zhang), `htang@ccny.cuny.edu` (H. S. Tang), `chengjian@buaa.edu.cn` (J. Cheng)

1 Introduction

In recent years, discontinuous Galerkin (DG) methods are gaining popularity in solving the incompressible Navier-Stokes (INS) equations [1–7]. The advantages of the DG methods over the traditional continuous finite element, finite difference and finite volume methods are well documented in the literatures [8–10]: the DG methods work well on arbitrary meshes, result in stable high order discretizations for both convective and diffusive operators, allow for a simple and unambiguous imposition of boundary conditions and well suit for parallelization and adaptivity.

One of the key ingredients of DG methods is the formulation of numerical interface fluxes, which provides a weak coupling of the unknowns from the target element and its surrounding neighbours. In a series of papers [11–13], Cockburn and coworkers provided and analyzed the local discontinuous Galerkin (LDG) method for the Stokes, Oseen and INS equations, respectively. For the Stokes system, they discretized the problem by using a fully discontinuous approach and proposed expressions of the numerical fluxes associated with the Laplacian and the incompressibility constraint [11]. For the Oseen equations, they provided a priori error estimation for the DG solution by treating the linear convective term with an upwinding scheme [12]. And for the nonlinear Navier-Stokes equations, in order to ensure that the DG formulation yields a locally conservative discretization, two approaches were proposed [13]. One approach is that they linearized the convective term and then used the results of [12] for the Oseen problem to prove the stability of the discrete solution. Through an iterative procedure they then recovered a locally conservative velocity field. In the other approach, the pressure p was replaced with the Bernoulli pressure $p+\frac{1}{2}|u|^2$; hence, local conservativity was attained. With this method, one can prove the boundedness of the approximate solution for the case of Dirichlet boundary conditions. For the case of outflow boundary conditions, however, this type of formulation leads to an unphysical solution at the outlet, as previously shown in the context of the continuous Galerkin approximation [14].

Shahbazi and coworkers [2] proposed a new high order DG discretization of the unsteady INS equations in convection-dominated flows using triangular and tetrahedral meshes. The scheme is based on a semi-explicit temporal discretization with explicit treatment of the nonlinear term and implicit treatment of the Stokes operator. The nonlinear term is discretized in divergence form by using the local Lax-Friedrichs fluxes. Thus, local conservativity, which is not offered by the LDG method, is inherent. However, due to the explicit treatment of the nonlinear term, time steps are strictly restricted to ensure stability and the computational efficiency are undermined.

Bassi et al. [1] relaxed the incompressibility constraint by adding an artificial compressibility term to the continuity equation to recover the hyperbolic nature of the INS equations. The inviscid numerical fluxes both in the continuity and in the momentum equation are computed using the values of velocity and pressure provided by the (exact) solution of the Riemann problem. For the cases of Stokes system and Oseen equations, the Riemann solver is resulted from a linear hyperbolic system, thus can be solved di-