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Global Mild Solutions and Long Time Decay of 3D Incompressible MHD Equations

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Abstract. We prove the global existence of a unique mild solutions to the incompressible MHD equations when the initial data are less than the viscosity coefficients in a new critical space introduced by Lei and Lin [1]. Moreover, we prove that solutions decay to zero as time goes to infinity.

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1 Introduction

We consider the three dimensional, incompressible magneto-hydrodynamical(MHD) equations in $\mathbb{R}^3$,

$$\begin{cases} \partial_t u + u\cdot\nabla u - \mu\Delta u + \nabla\pi = b\cdot\nabla b, \\ \partial_t b + u\cdot\nabla b - \eta\Delta b = b\cdot\nabla u, \\ \nabla\cdot u = \nabla\cdot b = 0, \\ (u,b)(x,0) = (u_0(x), b_0(x)). \end{cases} \tag{1.1}$$

Here u,b denote the flow velocity vector and the magnetic field vector respectively, p is a scalar pressure, $\mu>0$ is the kinematic viscosity and $\eta>0$ is the magnetic diffusivity, while (u_0,b_0) are the given initial data with $\nabla\cdot u_0 = \nabla\cdot b_0 = 0$.

System (1.1) has been extensively studied. Duraut and Lions [2] constructed a class of weak solutions with finite energy and a class of local strong solutions. However, whether this unique local solutions can exist globally and this weak solutions is unique are still

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outstanding challenging open problems. There have been many studies about these problems. One hand, the study of regularity criteria for weak solutions has been attempted by many researchers [3–12], on the other hand, the study of the global well-posedness under the special class of initial data is also an important way.

In the absence of magnetic field, system (1.1) becomes the incompressible Navier-Stokes (NS) equations. Recently, Lei and Lin [1] introduced a new function space

$$\chi^{-1}(\mathbb{R}^3):=\left\{f\in D'(\mathbb{R}^3),\int_{\mathbb{R}^3}|\xi|^{-1}|\hat{f}|\mathrm{d}\xi<+\infty\right\}$$

which is contained in BMO^{-1} and is equivalent to the Fourier-Herz space $\dot{B}_1^{-1}$. They proved that if the initial data $\|u_0\|_{\chi^{-1}}<\mu$, the NS equations admit a global unique mild solutions. In [13], the author considers the long time behavior of such solutions.

The aim of this paper is to extend the corresponding results available for 3D NS equations to the case of the MHD equations in the new space. We rely mainly on the method used in [1, 13]. However, we need more careful mathematical analysis due to magnetic field, and we give a blow-up criterion only in term of velocity field (see Remark 2.1). Let us mention that the well-posedness for the NS equations can be obtained by using Fixed Point Theorem (see [14]). However we cannot apply the method due to the presence of magnetic field.

Now, we state our main results about the global existence and decay properties of solutions to the system (1.1) in the following theorems:

Theorem 1.1. *The system* (1.1) *is well-posedness globally in time for the initial data in* $\chi^{-1}(\mathbb{R}^3)$ *satisfying*

$$\|u_0\|_{\chi^{-1}}+\|b_0\|_{\chi^{-1}}<\alpha<\min\{\mu,\eta\}, \tag{1.2}$$

Here α *is a positive constant. Moreover, the solutions* (u,b) *are in* $C(\mathbb{R}^+;\chi^{-1}(\mathbb{R}^3))\cap L^1(\mathbb{R}^+;\chi^1(\mathbb{R}^3))$, *and satisfying following inequality*

$$\begin{aligned}&\sup_{0\le t<\infty}\left[\|u(t)\|_{\chi^{-1}}+\|b(t)\|_{\chi^{-1}}+(\mu-\alpha)\int_0^t\|u(s)\|_{\chi^1}\mathrm{d}s+(\eta-\alpha)\int_0^t\|b(s)\|_{\chi^1}\mathrm{d}s\right]\\&\le\|u_0\|_{\chi^{-1}}+\|b_0\|_{\chi^{-1}}.\end{aligned}$$

Theorem 1.2. *Let* $(u,b)\in C(\mathbb{R}^+,\chi^{-1}(\mathbb{R}^3))$ *be global solutions of system* (1.1), *then*

$$\lim_{t\to+\infty}\sup(\|u(t)\|_{\chi^{-1}}+\|b(t)\|_{\chi^{-1}})=0.$$

Remark 1.1. In [13], the author pointed out that $\dot{H}^s(\mathbb{R}^3)\subseteq\chi^{-1}(\mathbb{R}^3),(s>\frac{1}{2})$. However, there is no comparison between $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ and $\chi^{-1}(\mathbb{R}^3)$ and they gave the following example:

$$f=F^{-1}\left(\frac{1}{|\xi|^2\ln|\xi|}1_{\{|\xi|>3\}}\right),\qquad g=F^{-1}\left(\sum_{n=1}^{\infty}\frac{1}{n}1_{|\xi-(2n+1)e_1|<1}\right).$$