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Existence and Regularity of a Weak Solution to a Class of Systems in a Multi-Connected Domain

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Abstract. We consider the existence and regularity of a weak solution to a class of systems containing a p-curl system in a multi-connected domain. This paper extends the result of the regularity theory for a class containing a p-curl system that is given in the author's previous paper. The optimal $C^{1+\alpha}$-regularity of a weak solution is shown in a multi-connected domain.

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1 Introduction

In this paper, we consider the existence and regularity of a weak solution to a class of systems containing a p-curl system in a bounded multi-connected domain Ω in $\mathbb{R}^3$.

In a bounded simply connected domain Ω in $\mathbb{R}^3$ without holes, Yin [1] considered the existence of a unique solution for the so-called p-curl system

$$\begin{cases} \mathrm{curl}\,[|\mathrm{curl}\,\boldsymbol{v}|^{p-2}\mathrm{curl}\,\boldsymbol{v}] = \boldsymbol{f} & \text{in } \Omega, \\ \mathrm{div}\,\boldsymbol{v} = 0 & \text{in } \Omega, \\ \boldsymbol{n}\times\boldsymbol{v} = \boldsymbol{0} & \text{on } \Gamma, \end{cases} \tag{1.1}$$

where Γ denotes the $C^{2+\alpha}$ $(\alpha\in(0,1))$ boundary of Ω, $p>1$, $\boldsymbol{n}$ the outer normal unit vector field to Γ, and $\boldsymbol{f}$ is a given vector field satisfying $\mathrm{div}\,\boldsymbol{f}=0$ in Ω. If $\boldsymbol{f}$ is a C^α-vector function, then he showed the optimal $C^{1+\beta}$-regularity for some $\beta\in(0,1)$ of a weak solution in Yin [2], see also Yin et al. [3].

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Eq. (1.1) is a steady-state approximation of Bean's critical state model for type II superconductors. For further physical background, see [3], Chapman [4] and Prigozhin [5].

Aramaki [6] extended the result of [2] on the $C^{1+\beta}$ regularity of a weak solution to a more general equation, in a simply connected domain without holes to the following system.

$$\begin{cases} \operatorname{curl}[S_t(x,|\operatorname{curl}\boldsymbol{v}|^2|)\operatorname{curl}\boldsymbol{v}]=\boldsymbol{f} & \text{in } \Omega, \\ \operatorname{div}\boldsymbol{v}=0 & \text{in } \Omega, \\ \boldsymbol{n}\times\boldsymbol{v}=\boldsymbol{0} & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where the function $S(x,t)\in C^2(\Omega\times(0,\infty))\cap C^0(\Omega\times[0,\infty))$ satisfies some structure conditions. Now and from now on, we denote $\frac{\partial}{\partial t}S(x,t)$ and $\frac{\partial^2}{\partial t^2}S(x,t)$ by $S_t(x,t)$ and $S_{tt}(x,t)$, respectively.

However, in a multi-connected domain, the systems (1.1) and (1.2) are not well posed. In fact, if the second Betti number is positive, for a weak solution $\boldsymbol{v}$ of (1.1) or (1.2), $\boldsymbol{v}+\boldsymbol{z}$, where $\boldsymbol{z}$ satisfies $\operatorname{curl}\boldsymbol{z}=\boldsymbol{0},\operatorname{div}\boldsymbol{z}=0$ in Ω and $\boldsymbol{z}\times\boldsymbol{n}=\boldsymbol{0}$ on Γ, is also a weak solution. Thus it is necessary to add some conditions to (1.1) and (1.2).

In this paper, we show the unique existence and optimal $C^{1+\beta}$-regularity of a weak solution to the system (1.2) with additive conditions.

The paper is organized as follows. In Section 2, we give some preliminaries and the main theorem. In Section 3, we give the existence of a weak solution of (2.10) below. Section 4 is devoted to the regularity of the weak solution obtained in Section 3.

2 Preliminaries and the main theorem

Since we allow that Ω is a multi-connected domain, we assume that Ω has the following conditions as in Amrouche and Seloula [7] (cf. Amrouche and Seloula [8], Dautray and Lions [9] and Girault and Raviart [10]). Let $\Omega\subset\mathbb{R}^3$ be a bounded domain of class $C^{2+\alpha}$ with the boundary Γ and Ω be locally situated on one side of Γ.

(1) Γ has a finite number of connected components $\Gamma_0,\Gamma_1,\ldots,\Gamma_m$ with Γ_0 denoting the boundary of the infinite connected component of $\mathbb{R}^3\setminus\overline{\Omega}$.

(2) There exist n connected open surfaces $\Sigma_j,(j=1,\ldots,n)$, called cuts, contained in Ω such that

  (a) Σ_j is an open subset of a smooth manifold $\mathcal{M}_j$.

  (b) $\partial\Sigma_j\subset\Gamma$ $(j=1,\ldots,n)$, where $\partial\Sigma_j$ denotes the boundary of Σ_j, and Σ_j is non-tangential to Γ.

  (c) $\overline{\Sigma_i}\cap\overline{\Sigma_j}=\emptyset(i\neq j)$.

  (d) The open set $\dot{\Omega}=\Omega\setminus(\cup_{i=1}^n\Sigma_i)$ is simply connected and pseudo $C^{1,1}$ class.