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ROBUST AND EFFICIENT MIXED HYBRID DISCONTINUOUS FINITE ELEMENT METHODS FOR ELLIPTIC INTERFACE PROBLEMS

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Abstract. Because of the discontinuity of the interface problems, it is natural to apply the discontinuous Galerkin (DG) finite element methods to solve those problems. In this work, both fitted and unfitted mixed hybrid discontinuous Galerkin (MHDG) finite element methods are proposed to solve the elliptic interface problems. For the fitted case, the problems can be solved directly by MHDG method. For the unfitted case, the *broken* basis functions (unnecessary to satisfy the jump conditions) are introduced to those elements which are cut across by interface, the weights depending on the volume fractions of cut elements and the different diffusions (or material heterogeneities) are used to stabilize the method, and the idea of the Nitsche's penalty method is applied to guarantee the jumps on the interface parts of cut elements. Unlike the immersed interface finite element methods (IIFEM), the two jump conditions are enforced weakly in our variational formulations. So, our unfitted interface MHDG method can be applied more easily than IIFEM to general cases, particularly when the immersed basis function cannot be constructed. Numerical results on convergence and sensitivities of both interface location within a cut element and material heterogeneities show that the proposed methods are robust and efficient for interface problems.

Key words. Elliptic interface problems, discontinuous Galerkin finite element methods, mixed and hybrid methods, Nitsche's penalty method, sensitivities of interface location and material heterogeneities.

1. Introduction

Interface problems arise frequently in many applications, as for example, in heat and mass transfer, electromagnetic wave propagation, cell and bubble formation, biological science, fluid mechanics and many other practical applications. In these problems, the solution and the flux are usually nonsmooth on interface. Interface problems with fixed interfaces can be solved efficiently by fitted interface methods [7, 5, 10, 20]. In these methods the meshes are constructed to align or approximate to the interface. However, for the moving interface problems, the fitted interface methods are very costly because of the generation of new fitted interface meshes at each time step.

To overcome this difficulty, the unfitted interface methods have been studied. The immersed boundary method was proposed in [29] to model blood flow in the heart. Since then, other unfitted interface methods, such as the immersed interface (finite difference) method [21, 19], the immersed interface finite element method (IIFEM) [23, 24, 15, 17, 18], the ghost fluid method [14, 25, 26], the extended finite element method (XFEM) [28, 4, 37], the Nitsche's penalty method [16, 1, 27], and so on, have been developed. In unfitted interface methods, the meshes are fixed, independent of the interface geometry and the interface usually cuts through cells. Then the moving interface problems can be solved with fixed meshes, without remeshing process.

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Because of the discontinuity of the interface problems, it is natural to apply the DG methods to solve those problems. The DG methods were introduced independently in [13, 31, 6]. Since then, numerous DG methods have been developed. Because of the flexibility for mesh and polynomial refinements, localizability, stability and parallelizability, the DG methods have been widely applied to many problems. Recently, a fitted DG method with a priori and a posteriori error estimations for the interface diffusion problem was studied in [10], the hybridizable DG (HDG) method based on [11] was applied to the fitted interface diffusion problem in [20], an unfitted DG method based on Nitsche's penalty method for the interface diffusion problem was introduced and analyzed in [27], and a selective immersed DG method for the interface diffusion problem was proposed in [17]. Besides, the mixed method can be used to get more precise approximation to the flux which is necessary in many applications, particularly for the coupled problems [34, 33, 35, 36, 32, 22]. From computational point of view, a particular advantage of the MHDG method is that it can be formulated and implemented at the element level. This allows to eliminate the primal and flux variables on the element level, then to obtain a global system only for the Lagrange multipliers.

In this work, we propose both fitted and unfitted MHDG methods for elliptic interface problems. For the fitted case, we solve the problems directly by the MHDG methos [13, 2, 8, 9, 11, 12]. For the unfitted case, similarly to the idea presented in[16], we propose the *broken* Raviart-Thomas basis functions (unnecessary to satisfy the jump conditions) to those elements which are cut across by interface, we introduce the weighted averages depending on the volume fractions of cut elements and the material heterogeneities to stabilize the method, and we apply the idea of the Nitsche's penalty method to guarantee the jumps on the interface parts of cut elements. Unlike the IIFEM method, the two jump conditions are enforced weakly in our variational formulations. Thus, our unfitted MHDG method can be applied more easily than IIFEM method to general cases, particularly when the immersed basis function cannot be constructed. Numerical results on convergence and sensitivities of both interface location within a cut element and material heterogeneities show that the proposed methods are robust and efficient for interface problems.

The paper is organized as follows. In section 2, we introduce the elliptic interface model problem, define the notations and the finite element spaces. In section 3, we present the fitted MHDG method and the corresponding numerical results. In section 4, we formulate the unfitted MHDG method. However, numerical results show that the flux on interface cannot be well approximated by the formulation. To get a good approximation to the flux on interface and to guarantee the interface jumps, we introduce two penalty terms to the formulation in section 4.2. As a result we obtain numerically a robust and efficient MHDG method for both cut elements with arbitrary small volume fractions and large material heterogeneities. Finally in section 5 we present some concluding remarks. The numerical analysis of the proposed interface MHDG method should be our next work.